A SIMPLE PROOF OF BALOG-SZEMERÉDI-GOWERS (ADAPTED FROM SCHOEN)

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Let

$$1_A \circ 1_B(x) = \sum_y 1_A(y) 1_B(x+y) = |A \cap (B+x)|.$$

Note that

$$\sum_s 1_A \circ 1_B(s) = |A|\,|B|$$

and

$$\sum_s 1_A \circ 1_B(s)^2 = E(A,B).$$

Lemma 1. *If $E(A,B) \geq K^{-1}|A|^2|B|$ then for any $0 < c < 1$ there is some $X \subseteq A$ such that $|X| \geq \frac{1}{2}K^{-1}|A|$ and for all but at most $c|X|^2$ many pairs $(a,b) \in X^2$,*

$$1_B \circ 1_B(a-b) \geq \frac{c}{2}K^{-2}|A|.$$

Proof. Let $A_s = A \cap (B+s)$. We have

$$\sum_s 1_A \circ 1_B(s)\,|A_s| = \sum_s 1_A \circ 1_B(s)^2.$$
$$= E(A,B).$$

By the Cauchy-Schwarz inequality therefore,

$$\sum_s 1_A \circ 1_B(s)\,|A_s|^2 \geq \frac{E(A,B)^2}{|A|\,|B|}.$$

For any $G \subseteq A^2$,

$$\sum_s 1_A \circ 1_B(s)\left|A_s^2 \cap G\right| = \sum_{(a,b)\in G}\sum_s 1_A \circ 1_B(s) 1_B(a-s) 1_B(b-s).$$

The innermost sum we bound using the trivial observation that $1_A \circ 1_B(s) \leq |B|$:

$$\sum_s 1_B \circ 1_B(s) 1_B(a-s) 1_B(b-s) \leq |B| \sum_x 1_B(a-x) 1_B(b-x)$$
$$= |B|\, 1_B \circ 1_B(a-b).$$

It follows that

$$\sum_s 1_A \circ 1_B(s)\left|A_s^2 \cap G\right| \leq |B| \sum_{(a,b)\in G} 1_B \circ 1_B(a-b).$$

In particular, if G is the set of pairs where

$$1_B \circ 1_B(a-b) \leq \frac{c}{2}\frac{E(A,B)^2}{|A|^3|B|^2},$$

then (using the trivial bound $|G| \leq |A|^2$)

$$\sum_s 1_A \circ 1_B(s)\left|A_s^2 \cap G\right| \leq \frac{c}{2}\frac{E(A,B)^2}{|A|\,|B|}.$$

Combining these inequalities (and using $\sum_s 1_A \circ 1_B(s) = |A|\,|B|$) we have

$$\sum_s 1_A \circ 1_B(s) \left(\frac{c}{2} \frac{E(A,B)^2}{|A|\,|B|} + \left|A_s^2 \cap G\right| \right) \leq \sum_s 1_A \circ 1_B(s) \left(c\,|A_s|^2 \right).$$

In particular there must exist some s such that, if $X = A_s$, then

$$\frac{c}{2} \frac{E(A,B)^2}{|A|^2\,|B|^2} + \left|X^2 \cap G\right| \leq c\,|X|^2.$$

In particular, such an X must satisfy

$$|X| \geq \frac{E(A,B)}{2^{1/2}\,|A|\,|B|}$$

and $\left|X^2 \cap G\right| \leq c\,|X|^2$, which is the statement of the lemma. □

Lemma 2 (Balog-Szemerédi-Gowers)**.** *If $E(A,B) \geq K^{-1}|A|^2|B|$ then there exists a subset $A' \subseteq A$ such that $|A'| \geq 2^{-4}K^{-1}|A|$ and*

$$|A' - A'| \leq 2^{10} \left(K\frac{|B|}{|A|} \right)^4 |A|.$$

Proof. We apply Lemma 1 with $c = 1/8$. Let $G \subseteq X^2$ be the set of pairs such that $1_B \circ 1_B(a-b) \geq 2^{-4}K^{-2}|A|$, so that $|G| \geq \frac{7}{8}|X|^2$. Let $A' \subseteq X$ be the set of $x \in X$ which are in at least $\frac{3}{4}|X|$ many pairs in G. Note that

$$\sum_{x \in X \setminus A'} \#\{(x,y) \in G : y \in X\} < \frac{3}{4}|X|^2$$

and hence

$$|X|\,|A'| \geq \sum_{x \in A'} \#\{(x,y) \in G : y \in X\} \geq \frac{1}{8}|X|^2,$$

so

$$|A'| \geq \frac{1}{8}|X| \geq 2^{-4}K^{-1}|A|.$$

We now claim that for any $x \in A' - A'$ there are $\geq 2^{-9}K^{-4}|A|^2|X|$ many quadruples $(a_1,a_2,a_3,a_4) \in B^4$ such that $x = a_1 - a_2 + a_3 - a_4$. Assuming this for the moment, since the total number of such quadruples is trivially at most $|B|^4$, we have

$$|B|^4 \geq |A' - A'|\,2^{-9}K^{-4}|A|^2|X|.$$

Recalling that $|X| \geq 2^{-1}K^{-1}|A|$, the result follows.

It remains to prove the claimed lower bound on the number of quadruples. Fix some $a, b \in A'$ such that $x = a - b$. By choice of A', there must be $\geq \frac{1}{2}|X|$ many $c \in X$ such that $(a,c), (b,c) \in G$, whence there are $\geq 2^{-4}K^{-2}|A|$ many $(a_1,a_2) \in B^2$ such that $a_1 - a_2 = a - c$, and similarly many $(a_3,a_4) \in B^2$ such that $a_3 - a_4 = b - c$. Any choice of such representations gives a quadruple such that $a_1 - a_2 - a_3 + a_4 = x$, since

$$x = a - b = (a-c) - (b-c) = (a_1 - a_2) - (a_3 - a_4).$$

Finally, note that different c must give rise to different quadruples, since a and b are fixed with x, and hence c can be recovered from the quadruple. There are $\geq \frac{1}{2}|X|$ many choices for c, and each c gives rise to $\geq 2^{-8}K^{-4}|A|^2$ many different quadruples, whence there are $\geq 2^{-9}K^{-4}|A|^2|X|$ many quadruples in total, as required. □