The Notorious Collatz conjecture

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The Collatz conjecture is one of the most elementary unsolved problems in mathematics.
It is also one of the most “dangerous” conjectures known – notorious for absorbing massive amounts of time from both professional and amateur mathematicians.
Introduced by Lothar Collatz in 1937, the conjecture is also known as the “3x+1 conjecture” or the “Syracuse problem”.
The conjecture involves an innocuous function $\text{Col}$ on the natural numbers $\{1,2,3,...\}$ defined by the following rule:

- $\text{Col}(n)$ equals $3n+1$ if $n$ is odd.
- $\text{Col}(n)$ equals $n/2$ if $n$ is even.
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Now consider iterates of the Collatz function $\text{Col}$, in which the output of the function is fed back into the input:

$\text{Col}^2(n) = \text{Col}(\text{Col}(n))$

$\text{Col}^3(n) = \text{Col}(\text{Col}(\text{Col}(\text{Col}(n))))$

etc.
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Every natural number $n$ generates a Collatz sequence (or Collatz orbit) $n, \text{Col}(n), \text{Col}^2(n), \text{Col}^3(n), \ldots$
For instance, \( n=1 \) generates the periodic Collatz sequence
\[ 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, \ldots \]
If a Collatz sequence reaches the value 1, it will then cycle through the values 1, 4, 2 indefinitely.
For instance, $n=6$ generates the Collatz sequence:
$6, 3, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, ...$
Collatz sequences are also known as hailstone sequences, as they can bounce up and down much like hailstones in a cloud were thought to.

For instance, \( n=27 \) generates the Collatz sequence

Hail too large for cloud to hold falls to earth causing strong cold downdraft

Hail growing in circulating convection currents

Freezing Level

Rain drops being sucked into the updraft
(Pedantic note: the modern theory of hailstone formation deviates from this classical model, being based instead on the properties of supercooled water droplets.)
But just as every hailstone eventually falls to the ground, we have the infamous Collatz conjecture. Every Collatz sequence eventually attains the value 1.
Despite hundreds of published papers on the Collatz conjecture, and many more unpublished works (including countless failed proofs), the conjecture remains unsolved today.
“Mathematics is not yet ripe enough for such questions.” – Paul Erdős, 1983

“This is an extraordinarily difficult problem, completely out of reach of present day mathematics.” – Jeff Lagarias, 2010

“For about a month everyone at Yale worked on it, with no result. A similar phenomenon happened when I mentioned it at the University of Chicago. A joke was made that this problem was part of a conspiracy to slow down mathematical research in the U.S.” – Shizuo Kakutani, 1960
THE COLLATZ CONJECTURE STATES THAT IF YOU PICK A NUMBER, AND IF IT'S EVEN DIVIDE IT BY TWO AND IF IT'S ODD MULTIPLY IT BY THREE AND ADD ONE, AND YOU REPEAT THIS PROCEDURE LONG ENOUGH, EVENTUALLY YOUR FRIENDS WILL STOP CALLING TO SEE IF YOU WANT TO HANG OUT.
The Collatz conjecture appears to be a mere mathematical curiosity, with no obvious real-world applications. Why should we try to solve it?
• Pure intellectual challenge
• A benchmark for testing our understanding of number theory
• Proof attempts have linked the problem to other areas of mathematics
• It is a simple, but non-trivial, toy model of a dynamical system
• Modest cash prizes ($50, Harold Coxeter; $500, Paul Erdős; £1000, Sir Bryan Thwaites)
• Bragging rights
Mathematically speaking, a (discrete) dynamical system is a state space $X$, together with a shift map $T$ from $X$ to itself. The iterates $T, T^2, T^3, \ldots$ describe the dynamics of the system.
In the Collatz dynamical system, the state space is the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and the shift map is the Collatz map $Col$. 
A sibling to the discrete dynamical systems are the continuous dynamical systems, where the dynamics are given by ordinary differential equations (ODE) or partial differential equations (PDE).
Many important real-world systems, such as fluids, ecosystems, and the climate, can be viewed as (continuous) dynamical systems.
The Collatz conjecture highlights the basic fact that even very simple equations can lead to amazingly complicated dynamics.
In mathematics, when we cannot solve a problem completely, we look for **partial results**. Even if they do not lead to a complete solution, they often reveal insights about the problem.
It is also useful to locate obstructions – such as counterexamples to related problems that highlight difficulties that have to be overcome in any proposed solution.
What partial results and obstructions do we have for the Collatz conjecture?
Partial result: in 2017, a distributed computing project verified the Collatz conjecture for all starting values $n$ up to $10^{20}$. So it is highly unlikely that a counterexample can be found just from pen and paper search.
One way the Collatz conjecture could fail is if there is a **cycle** – a Collatz sequence that repeats itself indefinitely – other than the known cycle 1, 2, 4, 1, 2, 4,... (or its shifts).
Partial result: it is known that any such cycle must have length at least 17,087,915. (Eliahou, 1993). So one cannot simply produce a short cycle to easily disprove the conjecture!
Obstruction: On the other hand, there are variants of the Collatz conjecture that have non-trivial cycles. For instance, if one modifies Col by sending an odd number $n$ to $3n-1$ rather than $3n+1$, then two additional cycles appear:

- $5, 14, 7, 20, 10, 5,\ldots$
- $17, 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 192, 91, 272, 136, 68, 34, 17,\ldots$

We don’t know if there are any further cycles for this map.
This obstruction shows that any proof of the Collatz conjecture must at some point use a property of the $3n+1$ map that is not shared by the $3n-1$ map.
Obstruction: the absence of non-trivial Collatz cycles can be shown to imply a difficult result in number theory:

**Theorem:** The gap between powers of 2 and powers of 3 goes to infinity.

\[ 3^2 - 2^3 = 9 - 8 = 1; \quad 2^5 - 3^3 = 32 - 27 = 5; \quad 2^8 - 3^5 = 256 - 243 = 13; \quad 3^7 - 2^{11} = 2187 - 2048 = 139; \ldots \]

Basically, if a power of 2 and power of 3 are too close together, they can be used to create a Collatz cycle.
**Theorem:** The gap between powers of 2 and powers of 3 goes to infinity.

This theorem is known to be true, but its proof is difficult, requiring a deep result known as **Baker’s theorem** (which earned Alan Baker the Fields medal in 1970).

So solving the Collatz conjecture may be at least as hard as proving Baker’s theorem!
One can try to work backwards and show that lots and lots of numbers get sent to 1 by the Collatz iteration.
Partial result: in 2003, Krasikov and Lagarias showed (with a computer-assisted proof) that for any large number $x$, there were at least $x^{0.84}$ initial values $n$ between 1 and $x$ whose Collatz iteration reached 1.
In 1987, John H. Conway invented a computer language called **FRACTRAN**, in which every program was a variant of the Collatz function $\text{Co}_\mathbb{N}$. The output of sequences could be used to perform mathematical computations!
For instance, the FRACTRAN program $Prime$ maps any natural number $n$ to the number $Prime(n)$, defined to equal

- $17n/91$ if $n$ is divisible by 91; else
- $78n/85$ if $n$ is divisible by 85; else
- $19n/51$ if $n$ is divisible by 51; else
- $23n/38$ if $n$ is divisible by 38; else
- $29n/33$ if $n$ is divisible by 33; else
- $77n/29$ if $n$ is divisible by 29; else
- $95n/23$ if $n$ is divisible by 23; else
- $77n/19$ if $n$ is divisible by 19; else
- $n/17$ if $n$ is divisible by 17; else
- $11n/13$ if $n$ is divisible by 13; else
- $13n/11$ if $n$ is divisible by 11; else
- $15n/2$ if $n$ is divisible by 2; else
- $n/7$ if $n$ is divisible by 7; else
- $55n$. 
Remarkable fact: the Prime orbit

\[ 2, \text{Prime}(2), \text{Prime}^2(2), \text{Prime}^3(2), \ldots \]

contains precisely the powers \(2^p\) of 2 whose exponents are primes (together with many non-powers of two). This FRACTRAN program computes primes!

In fact, FRACTRAN is Turing Complete. Roughly speaking, this means that any computation that can be performed by an ordinary computer, can also be computed by a FRACTRAN program!
Obstruction: There are FRACTRAN program sequences for which it is **undecidable** whether they will ever reach a certain target value $n_0$.

This is related to the **undecidability** of the **halting problem** for Turing Machines.
This *obstruction* demonstrates that there is NO general algorithm that can definitively resolve all questions resembling the Collatz conjecture.

Any solution to that conjecture must use special properties of the Collatz map $\text{Col}$ that are not shared by general FRACTRAN programs.
Partial result: we have a convincing (but non-rigorous) heuristic argument that predicts the truth of the Collatz conjecture.
The argument proceeds like this. The Collatz map $\text{Col}$ can take an odd number $n$ to a larger number $3n+1$. But this new number $3n+1$ is necessarily even, so the next application of $\text{Col}$ will divide it by 2.
Heuristically, there is a fifty-fifty chance that the number $(3n+1)/2$ will also be even, leading to further divisions by 2. Indeed, a probability theory calculation reveals that the "expected number" of divisions by 2 one experiences before reaching an odd number again is equal to two.

\[ n \rightarrow 3n+1 \rightarrow (3n+1)/2 \rightarrow (3n+1)/4 \]
As a consequence, if one starts with an odd number $n$, the next odd number in the Collatz sequence would be expected to equal approximately $3n/4$ on the average. Thus the average size of the odd numbers in the sequence will decrease towards 1, which supports the validity of the Collatz conjecture.
This heuristic also predicts that some variants of the Collatz map, such as the $5n+1$ map, will have orbits that go to infinity. This appears to be supported by numerics.

$n \rightarrow 5n+1 \rightarrow (5n+1)/2 \rightarrow (5n+1)/4$

7, 36, 18, 9, 46, 23, 116, 58, 29, 146, 73, 366, 183, 916, 458, 229, 1146, 573, 2866, 1433, 7166, 3583, 17916, ...
One can partially convert these heuristics into rigorous partial results by working statistically – studying the behavior of almost all Collatz orbits, rather than all orbits, thus excluding “outliers”.
in 1976, Terras showed that almost all initial values $n$ eventually iterated to a value less than $n$. (As a first approximation, think of “almost all” as meaning “at least 99.99% of all”.)
If one could show that all initial values $n$ (other than 1) iterated to something less than $n$, this would imply the Collatz conjecture by further iteration.
Partial result: Terras’s result was refined over the years. In 1979, Allouche showed that almost all initial values $n$ eventually iterated to a value less than $n^{0.869}$.
Partial result: in 1994, Korec lowered this bound further to $n^{0.7925}$. 

(almost all)  

$n$  ...  $< n^{0.7925}$
Partial result: In 2019, I showed that almost all initial values $n$ eventually iterated to a value less than $f(n)$, for any function $f()$ that grew to infinity, no matter how slowly. “Almost all Collatz orbits attain almost bounded values.”
For instance: almost all initial values $n$ eventually iterate to a value less than $\log(\log(\log(\log(\log(n))))).$
This is about as close as one can get to the Collatz conjecture without actually solving it.
Unfortunately, the statistical methods used in the proof seem to be unable to fully resolve the conjecture, which remains out of reach for now.
The argument was inspired by other dynamical systems results, and in particular by a 1994 result of Bourgain on constructing an invariant measure for the nonlinear Schrödinger equation.
A key difficulty with the Collatz iteration is that it can greatly distort the distribution of a set of numbers – some numbers collide into each other, others get skipped entirely.
As a consequence, the statistical behavior of Collatz iteration quickly becomes intractable to study.

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However, I was able to construct an (approximate) **invariant measure** – a distribution of numbers that iterates to something resembling a smaller version of itself.

(a certain variable number of iterations)

(plus or minus a small error)
Iterating this fact gives the result.

(after 49 pages of argument)
Thanks for listening!