1. The Proof

We will use the notation and results in [Tao14] leading up to Theorem 4.

**Theorem 1.1.** Theorem 4 of [Tao14] holds, i.e., any resolution of singularities $Y$ of $V$ is of general type.

**Proof.** First notice that since $V = H_1 \cap \cdots \cap H_4$ with each $H_i$ a degree two hypersurface we have, by (2.5), that

$$\omega_V \simeq \mathcal{O}_{\mathbb{P}^n}(1).$$

In particular, the global sections of $\omega_V$ correspond to the hyperplanes of the embedding $V \hookrightarrow \mathbb{P}^6$. In other words, this is the *canonical embedding* of $V$ (although this will not be used).

We want to prove that for a resolution of singularities $f : Y \to V$, the Kodaira dimension of $\omega_Y$ is maximal, i.e., that some power of $\omega_Y$ has enough sections to induce a generically finite map on $Y$. We will demonstrate there are enough global sections of $\omega_V$ that extend to global sections of $\omega_Y$ that they define a generically finite map on $Y$.

We will deal with the two types of singular points separately.

The rational double points are easy: By Fact 2(ii) they have the property that for $f : Y \to V$, $\omega_Y \supseteq f^*\omega_V$, which means exactly that all the global sections of $\omega_V$ extend to global sections of $\omega_Y$.

We would like to prove something similar for the elliptic points as well. This is a little trickier, because a priori differential forms pulled back from an elliptic singularity can pick up poles. By Example 2 these singularities are log canonical. Not all global sections of $\omega_V$ extend to $\omega_Y$, but we can extend those that vanish at the elliptic points by (2.7). In other words, let $E \subseteq Y$ denote the preimages via $f$ of the two points with elliptic (and hence log canonical) singularities. In other words, this is $\text{Ex}(f)$ if we restrict to neighbourhoods of these elliptic points. Since $\omega_V \simeq \mathcal{O}_{\mathbb{P}^n}(1)$, the global sections of the sheaf $\omega_V \otimes \mathcal{I}_{f(E)}$ can be identified by the set of all hyperplanes that vanish at these two points. Since these are log canonical singularities, the global sections of $\omega_V$ corresponding to these hyperplanes will extend to global sections of $\omega_Y$. So, all we need to prove is that these hyperplanes define a generically finite map on $V$. Collections of hyperplanes with preassigned basepoints (that is, points they must contain) correspond to projections to linear subspaces. In this case, the map defined by these hyperplanes is exactly the projection from the line connecting these two points to a complementary linear subspace isomorphic to $\mathbb{P}^4$. Looking at the explicit coordinates of the points we see that the line connecting them is given by $z = r_1 = r_2 = r_3 = r_4 = 0$ which.

*Date:* November 25, 2018.
means that the projection is given by
\[\begin{array}{ccc}
x & y & z \\
r_1 & r_2 & r_3 \\
r_4
d\end{array} \longrightarrow \begin{array}{ccc}
z & r_1 & r_2 \\
r_3 & r_4
d\end{array}.\]

We will prove that further projecting to the \([z : r_1] \text{ coordinates}\) the image of \(V\) is two-dimensional, which will complete the proof. Indeed, take an arbitrary pair of numbers \((r_1, r_2)\) and consider the quadric curves in \(\mathbb{P}^2\) defined by the equations \(Q_1\) and \(Q_2\) (defined below Claim 3 in [Tao14]). These curves intersect in a finite non-zero number of points (over an algebraically closed field, say \(\mathbb{C}\)). Notice that by the nature of the equations \(Q_1\) and \(Q_2\) implies that if \(r_1 \neq r_2\), then none of the solutions can have \(z = 0\). This means that the set
\[\{[1 : r_1 : r_2] \mid r_1 \neq r_2\}\]
is contained in the image of \(V\) and hence it has dimension two. \(\square\)

2. THE BACKGROUND

2.A. The canonical sheaf

Let \(U\) be a smooth (complex) variety and define its canonical sheaf, \(\omega_U\) as the sheaf of sections of the line bundle \((\det T_U)^*\), where \(T_U\) is the tangent bundle of \(U\).

As usual in algebraic geometry we will work with the sheaf, not the bundle. This has several advantages, for instance, this way the definition can be extended to singular varieties as follows.

Let \(X\) be an arbitrary variety which is non-singular in codimension one. In other words, assume that \(\text{codim}_X \text{Sing} X \geq 2\). For instance, let \(X\) be a surface with only isolated singular points. Now let \(\iota : U := (X \setminus \text{Sing} X) \hookrightarrow X\) denote the embedding of the non-singular locus of \(X\) into \(X\) and let
\[\omega_X := \iota_* \omega_U.\]

Here the operation \(\iota_*\) means the following: If \(\mathcal{F}\) is a sheaf on \(U\), then the sheaf \(\iota_* \mathcal{F}\), the push-forward of \(\mathcal{F}\), is defined by the following formula. For an open set \(V \subseteq X\), the sections of \(\iota_* \mathcal{F}\) on \(V\) are defined simply as the sections of \(\mathcal{F}\) on \(U \cap V\), i.e.,
\[\iota_* \mathcal{F}(V) := \mathcal{F}(U \cap V).\]

In other words, for an open set \(V \subseteq X\), we have
\[\omega_X(V) := \omega_U(U \cap V).\]

Essentially, we are considering differential forms on the non-singular locus, but think of them as formally extended to the singular points.

2.B. The adjunction formula

Let \(Z \subseteq U\) be a smooth closed subvariety of a smooth variety \(U\). Then we have the following short exact sequence of tangent and normal bundles on \(Z\):
\[0 \longrightarrow T_Z \longrightarrow T_U|_Z \longrightarrow N_{Z/U} \longrightarrow 0.\]

Taking determinants tells us that
\[\det T_U|_Z \cong \det T_Z \otimes \det N_{Z/U}.\]
Denoting the sheaf of sections of $N_{Z/U}$ by $\mathcal{N}_{Z/U}$ we obtain that
\begin{equation}
\omega_U|_Z \simeq \omega_Z \otimes (\det \mathcal{N}_{Z/U})^*.
\end{equation}

**Fact 1.** If $Z \subseteq U$ is locally defined by a single equation, then
\begin{equation}
\mathcal{N}_{Z/U}^* \simeq \mathcal{I}_Z|_Z,
\end{equation}
where $\mathcal{I}_Z \subseteq \mathcal{O}_U$ is the *ideal sheaf* of $Z$ in $U$, i.e., the sheaf of those local holomorphic functions on $U$ that vanish along $Z$, which by the assumption is a locally free sheaf of rank one, i.e., a sheaf of sections of a line bundle. In particular, in this case we have that
\begin{equation}
\omega_Z \simeq (\omega_U \otimes \mathcal{I}_Z^*)|_Z.
\end{equation}
This is called the *adjunction formula* and allows one to compute the canonical sheaf of complete intersections.

**Sketch of proof.** The proof is essentially the usual computation of the normal bundle of a hypersurface.

**Example 1.** Let $U = \mathbb{P}^n$ and $Z$ a hypersurface of degree $d$. Let $\mathcal{O}_{\mathbb{P}^n}(1)$ denote (the sheaf of sections of) the tautological bundle (there are two choices here which is usually opposite in topology and algebraic geometry. we choose the one that has non-zero sections.) We further define $\mathcal{O}_{\mathbb{P}^n}(-1) := \mathcal{O}_{\mathbb{P}^n}(1)^* = \mathcal{O}_{\mathbb{P}^n}(1)^{-1}$ and $\mathcal{O}_{\mathbb{P}^n}(d) := \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes d}$. In this terminology, it is relatively easy to see that
\[ \mathcal{I}_Z \simeq \mathcal{O}_{\mathbb{P}^n}(-d), \]
and hence the adjunction formula simplifies to
\[ \omega_Z \simeq (\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d)). \]
Taking determinants in the Euler sequence,
\[ 0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}^n}(1) \rightarrow T_{\mathbb{P}^n} \rightarrow 0, \]
tells us that
\[ \omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n - 1), \]
so, the above formula can be further simplified to
\[ \omega_Z \simeq (\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d)) \simeq \mathcal{O}_{\mathbb{P}^n}(d - n - 1). \]
Now let $X = H_1 \cap \cdots \cap H_r \subseteq \mathbb{P}^n$ be a complete intersection of hypersurfaces ("complete" means that $\dim X = n - r$). Let $d_i := \deg H_i$. Then repeatedly using the adjunction formula and the above computation tells us that
\begin{equation}
\omega_X \simeq \mathcal{O}_{\mathbb{P}^n} \left( \sum d_i - n - 1 \right).
\end{equation}
Notice that the due to our definition, these equalities remain true if $X$ is only non-singular in codimension one.
2.C. Singularities

2.C.1. Resolutions of singularities and exceptional sets. Let $X$ be an arbitrary variety. A resolution of singularities is a proper birational morphism $f : Y \to X$. A typical example is a composition of blow ups.

Given a resolution $f : Y \to X$, the exceptional set $\text{Ex}(f) \subseteq Y$ is the smallest (necessarily) closed subset of $Y$ such that $f$ is an isomorphism between $Y \setminus \text{Ex}(f)$ and $X \setminus f(\text{Ex}(f))$.

An interesting measure of how bad a singularity comes from comparing differential forms on a variety $X$ and its resolution. In general, it is always true that the pullback sheaf $f^*\omega_X$ agrees with $\omega_Y$ on $Y \setminus \text{Ex}(f)$, so the interesting question is whether sections in $f^*\omega_X$ (which are really differential forms on the non-singular part of $X$) extend to $\text{Ex}(f)$ as regular differentials, or if not, then what kind of poles they pick up.

Of course, the definition does not require that $X$ cannot be non-singular. It is a relatively easy exercise that if $X$ is non-singular, then it actually picks up zeros on the exceptional set. A baby case of this is that blowing up a smooth point on a surface means locally making a change of coordinates such as $(x, y)$ to $(x, z)$ where $y = xz$. In that case top differential forms are locally generated by $dx \wedge dy$. $f^*$ essentially making the substitution $y = xz$, so we get that $dx \wedge dy = dx \wedge d(xz) = xdx \wedge dz$. In these local coordinates, the exceptional set in the $(x, z)$ coordinates is defined by $x = 0$, so the above computation shows that indeed $f^*\omega_X$ will pick up a zero along the exceptional set.

2.C.2. Rational double points. These singularities are ubiquitous. This is probably due to the fact that many useful properties used to define singularities collide in this one notion in dimension two. Also, they are relatively easy to handle.

**Fact 2.** If $X$ has at worst rational double points then it has the following two properties:

(i) $\omega_X$ is a line bundle, and

(ii) for any resolution of singularities, $f : Y \to X$, $\omega_Y \supseteq f^*\omega_X$. Notice that since $Y$ is non-singular, this is meaningful even over singular points of $X$. In other words, this means that any $d$-form on $X$ (where $d = \dim X$) which is holomorphic (regular, if working over other fields than $\mathbb{C}$) in a (punctured) neighbourhood of a rational double point remains holomorphic on $Y$ in a neighbourhood of the preimage of that rational double point. In terms of the exceptional set discussed above, this means that it does not pick up poles along the exceptional set.

Furthermore, in dimension two, these two properties characterize rational double points. In higher dimensions these two conditions define Gorenstein canonical singularities.

2.C.3. Log canonical singularities. These are a bit more complicated, but in dimension two they are still quite manageable. The definition is essentially that these are singularities for which $f^*\omega_X$ picks up at worst simple poles when pulled back to a resolution of singularities.

For simplicity, let $X$ be such that $\omega_X$ is the sheaf of sections of a line bundle, e.g., $X$ is a complete intersection. Let $f : Y \to X$ be a resolution of singularities and let $E := \text{Ex}(f)$. Then $X$ has log canonical singularities if

\[
\omega_Y \supseteq f^*\omega_X \otimes \mathcal{I}_E,
\]

where $\mathcal{I}_E \subseteq \mathcal{O}_Y$ is the ideal sheaf of $E$ in $Y$. 
An alternative way to think about this is that those sections of $\omega_X$ that vanish along $f(E)$ extend to $Y$ without any poles. In a formula this would be saying that

$$\omega_Y \supseteq f^*(\omega_X \otimes \mathcal{I}_f(E)).$$

**Example 2.** A simple elliptic singularity, e.g., a cone over an elliptic curve is a log canonical singularity. This can be seen using the adjunction formula (2.4): Let $f: Y \to X$ be a resolution of singularities and let $E := \text{Ex}(f)$. Assume that $E$ is an irreducible elliptic curve. We know that $\omega_Y$ is isomorphic to $f^*\omega_X$ outside of $E$, and since those are both rank one locally free sheaves (sheaves of sections of line bundles), and so is the ideal sheaf $\mathcal{I}_E$ of $E$, we have that for some integer $a \in \mathbb{Z}$,

$$\omega_Y \simeq f^*\omega_X \otimes \mathcal{I}_E^a.$$

In order to prove that $X$ has log canonical singularities we need to prove that $a \leq 1$. In fact, we will prove that $a = 1$. By the adjunction formula (really the one before that) (2.2) we have that

$$\omega_Y|_E \simeq \omega_E \otimes \mathcal{N}_{E/Y}^* \simeq \mathcal{N}_{E/Y}^*.$$

$E$ is an elliptic curve, so $\omega_E \simeq \mathcal{O}_E$. (Also, we don’t need the determinant, because the normal bundle has rank one). On the other hand, $f(E)$ is a single point and $\omega_X$ is locally free, so it is trivial in a neighbourhood of $f(E)$. This means that $(f^*\omega_X)|_E \simeq \mathcal{O}_E$. Putting these together tells us that

$$\mathcal{I}_E^a|_E \simeq \mathcal{N}_{E/Y}^* \simeq \mathcal{I}_E|_E.$$ 

The second isomorphism follows from (2.3) and, of course, this implies that $a = 1$ as claimed. Note that this computation only used the fact that $\omega_E \simeq \mathcal{O}_E$, so it is valid for all cones over smooth projective varieties with a trivial canonical sheaf, e.g., abelian or Calabi-Yau varieties.

**References**

[Tao14] T. Tao: The Erdős-Ulam problem, varieties of general type, and the Bombieri-Lang conjecture,

University of Washington, Department of Mathematics, Seattle, WA 98195, USA

E-mail address: skovacs@uw.edu

URL: http://www.math.washington.edu/~kovacs