# Vaporizing and freezing the Riemann zeta function

Terence Tao

University of California, Los Angeles

Lezione Leonardesca, June 22, 2018

The Riemann zeta function $\zeta(s)$ is defined in the half-plane $\{\mathrm{Re}(s) > 1\}$ by the formula

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Because of the fundamental theorem of arithmetic, as well as the geometric series formula

$$\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots,$$

one has the Euler product formula

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

where the product is over primes. This links the zeta function to analytic number theory. The Euler product formula also shows that $\zeta(s) \neq 0$ whenever $\mathrm{Re}(s) > 1$.

Riemann also introduced the Riemann xi function

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s),$$

where $\Gamma$ is the gamma function

$$\Gamma(s) := \int_0^\infty e^{-t}t^s\frac{dt}{t}.$$

Initially, the xi function is only well defined in the region $\{\mathrm{Re}(s) > 1\}$. However, one can manipulate the formula for $\xi(s)$ in such a way that it extends to the whole complex plane as follows.

$$\xi(s) := \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s).$$

Using the standard formula $s\Gamma(s) = \Gamma(s+1)$ and some algebra, we have

$$\frac{s(s-1)}{2}\Gamma(\frac{s}{2}) = 2\Gamma(\frac{s+4}{2}) - 3\Gamma(\frac{s+2}{2})$$

and hence

$$\xi(s) = \sum_{n=1}^{\infty} 2\pi^{-s/2}n^{-s}\int_0^\infty e^{-t}t^{\frac{s+4}{2}}\frac{dt}{t} - 3\pi^{-s/2}n^{-s}\int_0^\infty e^{-t}t^{\frac{s+2}{2}}\frac{dt}{t}.$$

Dilating $t$ by $\pi n^2$, we can write this as

$$\xi(s) = \sum_{n=1}^{\infty} 2\pi^2 n^4 \int_0^\infty e^{-\pi n^2 t}t^{\frac{s+4}{2}}\frac{dt}{t} - 3\pi n^2 \int_0^\infty e^{-\pi n^2 t}t^{\frac{s+2}{2}}\frac{dt}{t}.$$

$$\xi(s) = \sum_{n=1}^\infty 2\pi^2 n^4 \int_0^\infty e^{-\pi n^2 t} t^{\frac{s+4}{2}} \frac{dt}{t} - 3\pi n^2 \int_0^\infty e^{-\pi n^2 t} t^{\frac{s+2}{2}} \frac{dt}{t}.$$

We make the change of variables $t = e^{4u}$ and use Fubini's theorem to arrive at the Fourier-Laplace representation

$$\xi(s) = 4 \int_{\mathbf{R}} \Phi(u) \exp(2su)\ du$$

where $\Phi$ is a relative of the theta function:

$$\Phi(u) := \sum_{n=1}^\infty (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}).$$

Making the renormalisation

$$H_0(z) := \frac{1}{8}\xi(\frac{1+iz}{2})$$

we then have the Fourier representation

$$H_0(z) = \frac{1}{2} \int_{\mathbf{R}} \Phi(u) \exp(izu)\ du.$$

It is clear that the function

$$\Phi(u) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) \exp(-\pi n^2 e^{4u}).$$

decays super-exponentially fast as $u \to +\infty$. From the Poisson summation formula one can obtain the functional equation

$$\Phi(-u) = \Phi(u),$$

and so one can now extend

$$H_0(z) = \frac{1}{2} \int_{\mathbf{R}} \Phi(u) \exp(izu)\, du$$

to the entire complex plane.

What do we know about $H_0$?

- It is an entire function (of order 1), obeying the functional equations $H_0(-z) = H_0(z)$ and $H_0(\overline{z}) = \overline{H_0(z)}$. Thus, the zeroes of $H_0$ are symmetric around the real and imaginary axes.
- All the zeroes of $H_0$ are contained in the strip $\{x + iy : |y| < 1\}$.
- Riemann hypothesis: All the zeroes of $H_0$ are real.

- Riemann-von Mangoldt formula: For $T \geq 2$, the number $N_0(T)$ of zeroes in the rectangle $\{x + iy : 0 \leq x \leq T; |y| \leq 1\}$ is $\frac{T}{4\pi} \log \frac{T}{4\pi} - \frac{T}{4\pi} + O(\log T)$. (Proven using upper and lower bounds on $H_0$ outside of the strip $\{x + iy : |y| < 1\}$, and upper bounds inside the strip.)
- A variant (due to Littlewood): If the Riemann hypothesis is true, then for any fixed $\alpha$, one has $N_0(T + \alpha) - N_0(T) = \alpha \log T + o(\log T)$ as $T \to \infty$.
- In particular, on RH, the mean spacing between zeroes of $H_0$ in $[T, 2T]$ is roughly $\frac{1}{\log T}$, and one has equidistribution of the zeroes at scales $\geq \eta(T)$ for some $\eta(T) = o(1)$.

We have the Riemann-Siegel approximation (or approximate functional equation), which says that in the regime where $y = O(1)$ and $x \gg 1$, one has

$$H_0(x+iy) \approx \frac{\pi^2}{\sqrt{8}} e^{-\pi x/8} \times$$

$$\left( e^{i\alpha_+} N^{\frac{7+y}{2}} \sum_{n=1}^{N} \frac{1}{n^{\frac{1+y-ix}{2}}} + e^{i\alpha_-} N^{\frac{7-y}{2}} \sum_{n=1}^{N} \frac{1}{n^{\frac{1-y+ix}{2}}} \right)$$

where $N := \lfloor \sqrt{\frac{x}{2\pi}} \rfloor$ and $\alpha_+, \alpha_-$ are the phases

$$\alpha_+ := -\frac{x}{4} \log \frac{x}{4\pi} + \frac{x}{4} + \frac{9-y}{8}\pi$$

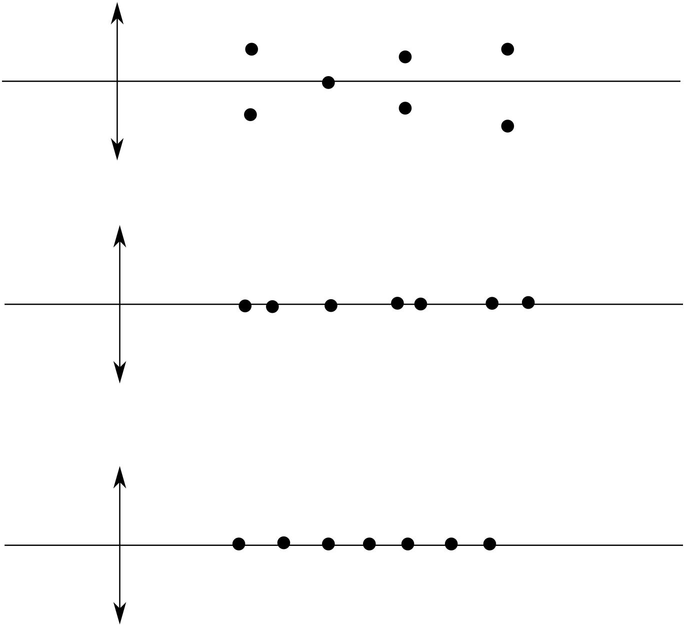
$$\alpha_- := \frac{x}{4} \log \frac{x}{4\pi} - \frac{x}{4} - \frac{9+y}{8}\pi.$$

In fact there are explicit formulae and good upper bounds on the error term in this approximation, making it well suited for numerics. The Riemann-Siegel approximation can be established by clever use of the residue theorem.

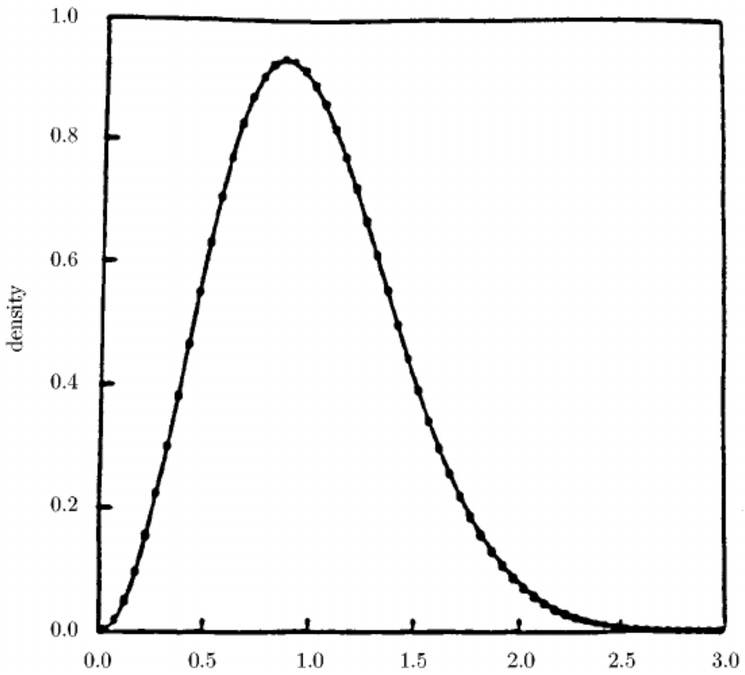
Informally, one can think of the zeroes of $H_0$ as living in one of three "states of matter":

- Gaseous state: the zeroes lie off of the real line.
- Liquid state: the zeroes lie on the real line, but are not close to being evenly spaced.
- Solid state: the zeroes lie on the real line, and are close to being evenly spaced.

The Riemann hypothesis can be viewed as an assertion that $H_0$ purely exists in the liquid and solid states, and never in the gaseous state.



There are some number-theoretic results which assert, roughly speaking, that $H_0$ is asymptotically not in the solid state (in contrast to, say, trigonometric functions or Bessel functions). For instance, in 1985, Conrey, Ghosh, Goldston, Gonek, and Heath-Brown showed (assuming the Riemann hypothesis) that a positive proportion of gaps $x_{n+1} - x_n$ between zeroes are less than 71% of the average spacing at that scale (which is $\frac{1+o(1)}{\log x_n}$). The GUE hypothesis in fact predicts that the zeta function is "highly liquid": normalised gaps $(x_{n+1} - x_n) \log x_n$ should be asymptotically distributed according to the Gaudin distribution, which allows for arbitrarily large or arbitrarily small normalised gaps.



density
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

In 1950, de Bruijn investigated what happens when one deforms the (renormalised) Riemann xi function

$$H_0(z) = \frac{1}{2} \int_{\mathbf{R}} \Phi(u) \exp(izu)\, du$$

in time to create the new functions

$$H_t(z) = \frac{1}{2} \int_{\mathbf{R}} e^{tu^2} \Phi(u) \exp(izu)\, du$$

for all $t \in \mathbf{R}$. These functions obey the backwards heat equation

$$\partial_t H_t(z) = -\partial_{zz} H_t(z).$$

It can be helpful to think of the functions $H_t$ as being increasingly "frozen" as time increases, or increasingly "vaporising" as time decreases. We say that the Riemann hypothesis holds at time $t$ if $H_t$ has all real zeroes. Of course we are most interested in this when $t = 0$!

Like $H_0$, the functions $H_t$ continue to be entire and obey the functional equations $H_t(-z) = H_t(z)$ and $H_t(\overline{z}) = \overline{H_t(z)}$. However, it no longer has an Euler product.

It follows from classical results of Pólya that if the Riemann hypothesis holds at some time $t_0$, then it holds at all later times $t > t_0$.

De Bruijn obtained the following improvement: if at time $t_0$ the zeroes of $H_{t_0}$ are contained in a horizontal strip $\{x + iy : |y| \leq y_0\}$, then at all later times $t > t_0$, they will be contained in a narrower strip $\{x + iy : |y| \leq \left(y_0^2 - 2(t - t_0)\right)_+^{1/2}\}$.

In particular, for the Riemann hypothesis will hold at time $t \geq t_0 + \frac{y_0^2}{2}$. The zeroes get attracted to the real axis as the Riemann xi function "freezes"!

Applying this with $t_0 = 0$ and $y_0 = 1$, de Bruijn concluded that the Riemann hypothesis was true for all times $t \geq \frac{1}{2}$.

One can explain this attraction to the real axis from an ODE perspective.

From tools such as the Hadamard factorisation theorem, one essentially has a product expansion of the form

$$H_t(z) \propto \prod_k (z - z_k(t))$$

where $z_k(t)$ are the zeroes of $H_t$, and one has to suitably renormalise the infinite product. Inserting this into the backwards heat equation, one eventually obtains the system of ordinary differential equations

$$\partial_t z_k(t) = 2 \sum_{j \neq k} \frac{1}{z_k(t) - z_j(t)}.$$

Informally: zeroes that are horizontally separated will repel each other, while zeroes that are vertically separated will attract each other. De Bruijn's theorem can then be explained by observing that any complex zero of $H_t$ is attracted to its complex conjugate.

In 1976, Newman showed that the Riemann hypothesis failed for sufficiently large negative $t$. Combining this with de Bruijn's results, we conclude that there exists a real number $-\infty < \Lambda \leq \frac{1}{2}$, now called the de Bruijn-Newman constant, such that the Riemann hypothesis at time $t$ is true if and only if $t \geq t_0$. The classical Riemann hypothesis is then equivalent to the assertion $\Lambda \leq 0$. Newman then made the opposite conjecture $\Lambda \geq 0$; intuitively, if the (classical) Riemann hypothesis is true, then it is only "barely so": any deformation of the Riemann xi function backwards in time destroys the Riemann hypothesis.

Known upper bounds on $\Lambda$:

- $\Lambda \leq \frac{1}{2}$ (de Bruijn, 1950)
- $\Lambda < \frac{1}{2}$ (Ki, Kim, Lee, 2009)
- $\Lambda \leq 0.22$ (`Polymath15`, 2018)

Known lower bounds on $\Lambda$:

- $\Lambda > -\infty$ (Newman, 1950)
- $\Lambda \geq -50$ (Csordas-Norfolk-Varga, 1988)
- $\Lambda \geq -5$ (te Riele, 1991)
- $\Lambda \geq -0.385$ (Norfolk-Ruttan-Varga, 1992)
- $\Lambda \geq -0.0991$ (Csordas-Ruttan-Varga, 1991)
- $\Lambda \geq -4.379 \times 10^{-6}$ (Csordas-Smith-Varga, 1994)
- $\Lambda \geq -5.895 \times 10^{-9}$ (Csordas-Odlyzko-Smith-Varga, 1993)
- $\Lambda \geq -2.63 \times 10^{-9}$ (Odlyzko, 2000)
- $\Lambda \geq -1.15 \times 10^{-11}$ (Saouter-Gourdon-Demichel, 2011)
- $\Lambda \geq 0$ (Rodgers-T., 2018)

Previous lower bounds on $\Lambda$ proceeded, roughly speaking, by running the law of motion of zeroes

$$\partial_t z_k(t) = 2 \sum_{j \neq k} \frac{1}{z_k(t) - z_j(t)}$$

backwards in time, so that zeroes on the real line now attract each other instead of repelling.
One then numerically locates a Lehmer pair - two zeroes of $H_0$ that are unusually close to each other. Going backwards in time, these zeroes quickly collide and then bounce off into the complex plane, "vaporising" this portion of the function from the liquid state to the gaseous state.

The method of proof of Newman's conjecture $\Lambda \geq 0$ proceeds, roughly speaking, as follows.

- Assume for contradiction that $\Lambda < 0$, in particular the Riemann hypothesis is true at times $-\Lambda \leq t \leq 0$. Thus $H_t$ is liquid or solid at these times.
- Show that for any $\varepsilon > 0$, the asymptotic time to relaxation to equilibrium is less than $\varepsilon$: if $H_{t_0}$ is asymptotically liquid or solid, then $H_{t_0+\varepsilon}$ is asymptotically solid.
- From the results of Conrey, Ghosh, Goldston, Gonek, and Heath-Brown mentioned earlier, $H_0$ is not asymptotically solid.
- Setting $t_0 = \Lambda$ and $\varepsilon = -\Lambda$, we obtain a contradiction.

The equilibrium states for the ODE

$$\partial_t z_k(t) = 2 \sum_{j \neq k} \frac{1}{z_k(t) - z_j(t)}$$

occur when the $z_k$ are arranged in an infinite arithmetic progression; at the level of the backwards heat equation, this corresponds to equilibrium solutions such as $H_t(z) = C \cos(\omega z + \theta)$, whose zeroes are a fixed arithmetic progression of spacing $\frac{\pi}{\omega}$.

The main difficulty is then to rigorously establish asymptotic local convergence to equilibrium in arbitrarily small time scales.

The approach was inspired by recent analogous local convergence to equilibrium results in random matrix theory by Erdős, Schlein, and Yau. It is based on exploiting the monotonicity and convexity properties of the entropy functional

$$\mathrm{H}(t) := \sum_{j \neq k} \log \frac{1}{|z_k(t) - z_j(t)|}.$$

Formally, the zeroes $z_k(t)$ evolve by the gradient flow for this functional, and the functional is formally decreasing and convex in time, suggesting convergence to equilibrium.

Unfortunately, this functional is actually infinite, but one can work with suitable truncations and renormalisations of this functional.

In order to keep the time to relaxation under $\varepsilon$, it was necessary to prove a generalisation of the Riemann-von Mangoldt formula: for any $-\Lambda \leq t \leq 0$ and $T \gg 1$, the number $N_t(T)$ of zeroes $H_t(x+iy) = 0$ with $0 \leq x \leq T$ is equal to $\frac{T}{4\pi} \log \frac{T}{4\pi} - \frac{T}{4\pi} + O(\log^2 T)$, and for any $\alpha > 0$, we have the variant $N_t(T + \alpha \log T) - N_t(T) = \alpha \log^2 T + o(\log^2 T)$.

The bounds here are worse than those for $N_0$ because the Euler product is no longer available to bound $H_t$ away from zero.

Now we turn to the problem of upper bounding $\Lambda$. The first step is to generalise the Riemann-Siegel approximation

$$H_0(x+iy) \approx \frac{\pi^2}{\sqrt{8}} e^{-\pi x/8} \times$$

$$\left( e^{i\alpha_+} N^{\frac{7+y}{2}} \sum_{n=1}^{N} \frac{1}{n^{\frac{1+y-ix}{2}}} + e^{i\alpha_-} N^{\frac{7-y}{2}} \sum_{n=1}^{N} \frac{1}{n^{\frac{1-y+ix}{2}}} \right).$$

Using a lot of contour shifting (the saddle point method), we obtained the generalised Riemann-Siegel approximation

$$H_t(x+iy) \approx \frac{\pi^2}{\sqrt{8}} e^{-\pi x/8} \exp\left(\frac{t}{16}\left(\log^2 \frac{x}{4\pi} - \frac{\pi^2}{4}\right)\right) \times$$

$$\left( \sum_{n=1}^{N} e^{i\alpha_+(t)} N^{\frac{7+y}{2}} \frac{1}{n^{\frac{1+y-ix}{2} + \frac{t}{4}\log\frac{N^2}{n} - \frac{\pi i t}{8}}} + e^{i\alpha_-(t)} N^{\frac{7-y}{2}} \frac{1}{n^{\frac{1-y+ix}{2} + \frac{t}{4}\log\frac{N^2}{n} + \frac{\pi i t}{8}}} \right)$$

for $0 \leq t \ll 1$, $y = O(1)$, $x \gg 1$, where

$$\alpha_+ := -\frac{x}{4}\log\frac{x}{4\pi} + \frac{x}{4} + \frac{9-y}{8}\pi + \frac{t}{32}\log\frac{x}{4\pi}$$

$$\alpha_- := \frac{x}{4}\log\frac{x}{4\pi} - \frac{x}{4} - \frac{9+y}{8}\pi + \frac{t}{32}\log\frac{x}{4\pi}.$$

For large $x$ ($x \geq \exp(C/t)$), the approximation "freezes" to

$$H_t(x+iy) \approx \frac{\pi^2}{\sqrt{8}} e^{-\pi x/8} \exp(\frac{t}{16}(\log^2 \frac{x}{4\pi} - \frac{\pi^2}{4}))\times$$

$$\left(e^{i\alpha_+(t)} N^{\frac{7+y}{2}} + e^{i\alpha_-(t)} N^{\frac{7-y}{2}}\right).$$

In particular, the zeroes "solidify" to the real axis, close to the set where $\alpha_+(t) - \alpha_-(t)$ is a multiple of $\pi$, and the Riemann hypothesis is provably true in this region. For smaller $x$ ($x < \exp(C/t)$), one could still have "liquid" or "gaseous" behavior. But this is just a finite region that can be checked numerically for a given value of $t$!

For fixed values of $t$ (e.g. $t = 0.22$), the remaining numerical calculation is still somewhat prohibitive. We were able to take a numerical shortcut by exploiting the existing extensive work on numerical verification of the Riemann hypothesis, which ensures that all zeroes of $H_0$ up to a very large value of $x$ lie on the real line. The main difficulty is then to erect a numerically verifiable "barrier" that ensures that as time increases, no zeroes of large imaginary part enter this region. This is done by use of the argument principle.

Thanks for listening!