Let $1/4 \leq \varepsilon \leq 1/2$. All variables $x, y, z$ are assumed to be non-negative by default. Let $M = \hat{M}_{3, \varepsilon, 1}^n$ be the best constant for which one has the inequality

\begin{equation}
3 \int \int_{x+y \leq 1-\varepsilon} \left( \int F(x, y, z) \, dz \right)^2 \, dxdy \leq M \int_R F(x, y, z)^2 \, dxdydz \tag{0.1}
\end{equation}

whenever $F$ is supported on $R := \{(x, y, z) : x + y + z \leq 3/2\}$, is symmetric, and obeys the vanishing marginal condition

\begin{equation}
\int F(x, y, z) \, dz = 0 \text{ whenever } x + y \geq 1 + \varepsilon. \tag{0.2}
\end{equation}

**Proposition 0.1** (Adjoint formulation). $M$ is also the best constant for which one has the inequality

\begin{equation}
\int_R (G(x, y) + G(y, z) + G(z, x))^2 \, dxdydz \leq 3M \int \int_{x+y \leq 1-\varepsilon} G(x, y)^2 \, dxdy \tag{0.3}
\end{equation}

whenever $G$ is symmetric, supported on $\{(x, y) : x + y \leq 1 - \varepsilon\} \cup \{(x, y) : 1 + \varepsilon \leq x + y \leq 3/2\}$, and the function $F(x, y, z) := G(x, y) + G(y, z) + G(z, x)$ obeys the vanishing marginal condition \((0.2)\).

**Proof.** Let us first show that if $M$ is such that \((0.1)\) holds, then \((0.3)\) also holds. Let $G$ be as in \((0.3)\), and set $F(x, y, z) := G(x, y) + G(y, z) + G(z, x)$. Then by symmetry,
Fubini, (0.2), the support of $G$, Cauchy-Schwarz, and (0.1), one has

$$\int_{R} (G(x,y) + G(y,z) + G(z,x))^2 \, dx dy dz$$

$$= \int_{R} (G(x,y) + G(y,z) + G(z,x)) F(x,y,z) \, dx dy dz$$

$$= 3 \int_{R} G(x,y) F(x,y,z) \, dx dy dz$$

$$= 3 \int_{x+y\leq 1/2} G(x,y) (\int F(x,y,z) \, dz) \, dx dy$$

$$= 3 \int_{x+y\leq 1-\varepsilon} G(x,y) (\int F(x,y,z) \, dz) \, dx dy$$

$$\leq (3) \int_{x+y\leq 1-\varepsilon} G(x,y)^2 \, dx dy)^{1/2} (3 \int_{x+y\leq 1-\varepsilon} (\int F(x,y,z) \, dz)^2 \, dx dy)^{1/2}$$

$$\leq (3) \int_{x+y\leq 1-\varepsilon} G(x,y)^2 \, dx dy)^{1/2} (M \int_{R} F(x,y,z)^2 \, dx dy dz)^{1/2}$$

$$= (3) \int_{x+y\leq 1-\varepsilon} G(x,y)^2 \, dx dy)^{1/2} (M \int_{R} (G(x,y) + G(y,z) + G(z,x))^2 \, dx dy dz)^{1/2},$$

and (0.3) follows.

Conversely, suppose that $M$ is such that (0.3) holds; we now show (0.1). Let $F$ be as in (0.1), and let $G_0$ be defined by setting $G_0(x,y) := \int F(x,y,z) \, dz$ when $x+y \leq 1-\varepsilon$, and $G_0(x,y) = 0$ otherwise. Let $G_1$ be a symmetric function supported on $\{1+\varepsilon \leq x+y \leq 3/2\}$ to be chosen later, and set $G := G_0 + G_1$ and $\tilde{F}(x,y,z) := G(x,y) + G(y,z) + G(z,x)$. We can choose $G_1$ so that the function $\tilde{F}$ obeys the vanishing marginal condition (0.2); indeed, in the symmetric functions in $L^2(R)$, one may verify that the class of $F$ obeying (0.2) is nothing other than the orthogonal complement of the functions of the form $G_1(x,y) + G_1(y,z) + G_1(z,x)$ with $G_1$ supported on $\{1+\varepsilon \leq x+y \leq 3/2\}$. By (0.2),
symmetry, Cauchy-Schwarz, and (0.3) one has

\begin{align*}
3 \int_{x+y \leq 1-\varepsilon} \left( \int G(x, y, z) \, dz \right)^2 \, dxdy & \\
& = 3 \int_{x+y \leq 1-\varepsilon} \left( \int G(x, y, z) \, dz \right) G_0(x, y) \, dxdy \\
& = 3 \int_R F(x, y, z) G_0(x, y) \, dxdydz \\
& = 3 \int_R F(x, y, z) G(x, y) \, dxdydz \\
& = \int_R F(x, y, z) (G(x, y) + G(y, z) + G(z, x)) \, dxdydz \\
& \leq (\int_R F(x, y, z)^2 \, dxdydz)^{1/2} (\int_R (G(x, y) + G(y, z) + G(z, x)^2 \, dxdydz)^{1/2} \\
& \leq (\int_R F(x, y, z)^2 \, dxdydz)^{1/2} (3M \int_{x+y \leq 1-\varepsilon} G(x, y)^2 \, dxdy)^{1/2} \\
& = (\int_R F(x, y, z)^2 \, dxdydz)^{1/2} (3M \int_{x+y \leq 1-\varepsilon} (\int F(x, y, z) \, dz)^2 \, dxdy)^{1/2}
\end{align*}

and (0.1) follows. \hfill \square

We can simplify the left-hand side of (0.3) further, using the vanishing marginal condition (0.2). Introduce the inner product

\[ \langle G, G' \rangle := \int_R (G(x, y) + G(y, z) + G(z, x))(G'(x, y) + G'(y, z) + G'(z, x)) \, dxdydz \]

then the left-hand side of (0.3) is

\[ \langle G, G' \rangle = \langle G, G_1 \rangle + \langle G, G_0 \rangle. \]

On the other hand, the vanishing marginal condition (0.2) tells us that \( \langle G, G_1 \rangle = 0 \). Thus

\[ \langle G, G \rangle = \langle G, G_0 \rangle. \]

We thus see that

\[ M = \sup \frac{\langle G, G_0 \rangle}{3 \int G_0^2 \, dxdy} \]

whenever \( G_0, G_1 \) are symmetric and supported on \( \{ x+y \leq 1-\varepsilon \} \) and \( \{ 1+\varepsilon \leq x+y \leq 3/2 \} \) respectively, and the function

\[ F(x, y, z) = G(x, y) + G(y, z) + G(z, x) \]

with \( G = G_0 + G_1 \) obeys (0.2).
We simplify things further. Observe that
\[
\langle G, G_0 \rangle = \int_R (G(x, y) + G(y, z) + G(z, x))(G_0(x, y) + G_0(y, z) + G_0(z, x)) \, dxdydz
\]
\[
= 3 \int_R (G(x, y) + G(y, z) + G(z, x))G_0(x, y) \, dxdydz
\]
\[
= 3 \int_R G_0(x, y) \, dxdydz + 6 \int_R G(z, x)G_0(x, y) \, dxdydz
\]
\[
= 3 \int_{x+y \leq 1-\varepsilon} G_0(x, y)^2(3/2 - x - y) \, dxdy + 6 \int_{x+y \leq 1-\varepsilon} G_0(x, y)(\int_0^{3/2-x-y} G(z, x) \, dz) dxdy
\]
\[
= 3 \int_{x+y \leq 1-\varepsilon} G_0(x, y)^2(3/2 - x - y) \, dxdy + 6 \int_{x+y \leq 1-\varepsilon} G_0(x, y)(\int_0^{1-\varepsilon-x} G(z, x) \, dz) dxdy
\]
\[
+ 6 \int_{x+y \leq 1-\varepsilon; y \leq 1-2\varepsilon} G_0(x, y)(\int_{1+\varepsilon-x}^{3/2-x-y} G_1(z, x) \, dz) dxdy
\]
\[
= 3 \int_{x+y \leq 1-\varepsilon} G_0(x, y)^2(3/2 - x - y) \, dxdy + 6 \int_{x \leq 1-\varepsilon} G_0(x, y)(\int_0^{1-\varepsilon-x} G_0(x, y) \, dy)^2 \, dx
\]
\[
+ 6 \int_{1+\varepsilon \leq x + z \leq 3/2} G_1(x, z)(\int_{y \leq 3/2-x-z} G_0(x, y) \, dy) \, dx \, dz.
\]
Here we use that \( \varepsilon \geq 1/4 \) to ensure that \( 1 - \varepsilon - x \leq 3/2 - x - y \) whenever \( x + y \leq 1 - \varepsilon \). Thus
\[
M = \sup J I
\]
where
\[
J := \int_{x+y \leq 1-\varepsilon} G_0(x, y)^2(3/2 - x - y) \, dxdy + 2 \int_{x \leq 1-\varepsilon} (\int_0^{1-\varepsilon-x} G_0(x, y) \, dy)^2 \, dx
\]
\[
+ 2 \int_{1+\varepsilon \leq x + z \leq 3/2} G_1(x, z)(\int_{y \leq 3/2-x-z} G_0(x, y) \, dy)
\]
and
\[
I := \int \int G_0^2 \, dxdy.
\]
Now we work on \( G_1 \). From (0.2) we see that
\[
\int_0^{3/2-x-y} G(x, y) + G(y, z) + G(z, x) \, dz = 0
\]
whenever \( x + y \geq 1 + \varepsilon \), thus
\[
G_1(x, y)(3/2 - x - y) + \int_0^{3/2-x-y} G(y, z) \, dz + \int_0^{3/2-x-y} G(z, x) \, dz = 0. \tag{0.4}
\]
At this point, we set \( \varepsilon = 1/4 \), and introduce the large triangular region
\[
A := \{ x + y \leq 1 - 3/4 \}
\]
(that $G_0$ is supported on), and the smaller triangular and trapezoidal regions

$$B := \{x + y \leq 3/2; y \geq 5/4\}$$
$$C := \{x + y \geq 5/4; y \leq 5/4; x \leq 1/4\}$$
$$D := \{5/4 \leq x + y \leq 3/2; x \geq 1/4; y \geq 3/4\}$$
$$E := \{x + y \geq 5/4; x, y \leq 3/4\}$$

which cover a bit more than half of the support of $G_1$. Let $G_{1,B}, G_{1,C}, G_{1,D}, G_{1,E}$ be the restrictions of $G_1$ to $B, C, D, E$ respectively; we assume these to be polynomial, as we do with $G_0$. We also introduce the average

$$H(x, w) := \frac{1}{w} \int_0^w G_0(x, y) \, dy$$

(0.5) of $G_0$; note that $H$ is still a polynomial on $A$ if $G_0$ is, although $H$ is no longer symmetric. The vanishing moment condition (0.4) then becomes

$$(3/2 - x - y)G_{1,B}(x, y) + \int_0^{3/2 - x - y} G_{1,B}(y, z) \, dz + (3/2 - x - y)H(x, 3/2 - x - y) = 0$$

(0.6) on $B$,

$$(3/2 - x - y)G_{1,C}(x, y) + \int_{3/4 - y}^{3/2 - x - y} G_{1,C}(y, z) \, dz + (3/2 - x - y)H(x, 3/2 - x - y) = 0$$

(0.7) on $C$, 

$$(3/2 - x - y)G_{1,D}(x, y) + 0 + (3/2 - x - y)H(x, 3/2 - x - y) = 0$$

(0.8) on $D$, and

$$(3/2 - x - y)G_{1,E}(x, y) + (3/2 - x - y)H(y, 3/4 - y) + (3/2 - x - y)H(x, 3/4 - x) = 0$$

(0.9) on $E$. Thus we have exact formulae

$$G_{1,B}(x, y) = -H(x, 3/2 - x - y)$$

(0.10)

and

$$G_{1,E}(x, y) = -H(y, 3/4 - y) - H(x, 3/4 - x)$$

(0.11) on $D, E$ respectively, and linear constraints connecting the coefficients of $G_{1,B}, G_{1,C}$ with the coefficients of $H$ (which are defined in terms of $G_0$).
We then have

\begin{align*}
J & = \int_{x+y \leq 3/4} G_0(x, y)^2 (3/2 - x - y) \, dx \, dy \\
& \quad + 2 \int_{x \leq 3/4} \left( \int_0^{3/4-x} G_0(x, y) \, dy \right)^2 dx \\
& \quad + 2 \int_B G_{1,B}(x, z)(3/2 - x - z)H(x, 3/2 - x - z) \, dx \, dz \\
& \quad + 2 \int_C G_{1,C}(x, z)(3/2 - x - z)H(x, 3/2 - x - z) \, dx \, dz \\
& \quad + 2 \int_D G_{1,D}(x, z)(3/2 - x - z)H(x, 3/2 - x - z) \, dx \, dz \\
& \quad + 2 \int_E G_{1,E}(x, z)(3/4 - x)H(x, 3/4 - x) \, dx \, dz
\end{align*}

and

\begin{align*}
I & = \int_{x+y \leq 3/4} G_0(x, y)^2 \, dx \, dy.
\end{align*}

We thus have

\begin{align*}
M = \sup I
\end{align*}

where \( G_0, G_{1,B}, G_{1,C} \) range over all polynomials of two variables obeying the constraints (0.6), (0.7), where \( H, G_{1,D}, G_{1,E} \) are defined by (0.5), (0.10), (0.11), with \( G_0 \) symmetric. If we make \( G_0, G_{1,B}, G_{1,C} \) be of degree up to \( D \), then there are about \( 5D^2/2 \) degrees of freedom with \( 2D^2 \) constraints.

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