NOTES ON ZHANG'S PRIME GAPS PAPER

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1. Zhang's results

For any natural number H, let $P(H)$ denote the assertion that there are infinitely many pairs of distinct primes p, q with $|p-q| \leq H$; thus for instance $P(2)$ is the twin prime conjecture. In [5], Zhang established for the first time a result of the form $P(H)$ for some finite H:

Theorem 1. [5] $P(7 \times 10^7)$ *is true.*

This result is deduced from the following result. Call an *admissible set* to be a finite set of integers $\mathcal{H}$ which avoids at least one residue class modulo p for each prime p. For any natural number k_0, let $Q(k_0)$ denote the assertion that for any admissible set $\mathcal{H}$ of integers of cardinality k_0, there are infinitely many translates $n+\mathcal{H}$ of $\mathcal{H}$ that contain at least two primes. Note that if $\mathcal{H}$ is an admissible set of cardinality k_0, then $Q(k_0)$ implies $P(\mathrm{diam}(H))$. Theorem 1 is then derived from Theorem 2 below, together with a construction of an admissible set of cardinality 3.5×10^6 and diameter at most 7×10^7:

Theorem 2. [5, Theorem 1] $Q(3.5 \times 10^6)$ *is true.*

One can improve Theorem 1 by constructing narrower admissible sets $\mathcal{H}$ of the specified cardinality 3.5×10^6; in particular one can show $P(57, 554, 086)$ in this fashion [2], by selecting a set $\mathcal{H}$ of the form

$$\mathcal{H} := \{\pm 1, \pm p_m, \ldots, \pm p_{m+k_0/2-1}\}$$

with $k_0 := 3.5 \times 10^6$ and $m := 36, 716$, with p_n denoting the n^{th} prime; this construction first appears in [4].

Theorem 2 is in turn primarily deduced from a deep improvement of the Bombieri-Vinogradov inequality, which we now pause to state. For any $\varpi > 0$, let $R(\varpi)$ denote the assertion that the estimate

$$\sum_{d<D^2; d|\mathcal{P}} \sum_{c \in h_0+\mathcal{C}(d)} |\Delta(\theta; d, c)| \ll x \log^{-A} x$$

for all admissible tuples $\mathcal{H}$, all $h_0 \in \mathcal{H}$, all fixed $A > 0$ and all sufficiently large x, where

$$D := x^{1/4+\varpi}$$
$$D_1 := x^{\varpi}$$
$$\mathcal{P} := \prod_{p<D_1} p,$$

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and for each positive integer d, $\mathcal{C}(d)$ is the set of residue classes $c \mod d$ coprime to d such that $\prod_{h \in \mathcal{H}}(c+h) = 0 \mod d$. Also, $\theta : \mathbf{N} \to \mathbb{R}$ is the function with $\theta(n) := \log n$ when n is prime, and $\theta(n) = 0$ otherwise, and

$$\Delta(\theta; d, c) := \sum_{x \leq n \leq 2x : n = c \mod d} \theta(n) - \frac{1}{\phi(n)} \sum_{x \leq n \leq 2x; (n,d)=1} \theta(n)$$

is the error term in the prime number theorem in arithmetic progressions for $c \bmod d$. Here the implied constant can depend on $\mathcal{H}, A$ but is independent of x.

Theorem 3. [5, Theorem 2] *$R(\frac{1}{1168})$ is true.*

Any result of the form $R(\varpi)$ for some $\varpi > 0$ implies a result of the form $Q(k_0)$ for some $k_0 < \infty$. We review this argument (a form of which already appears in [3]) in Section 2.

2. Deducing Theorem 2 from Theorem 3

We now recall how $R(\varpi)$ for some $\varpi > 0$ can be used to establish $Q(k_0)$ for some $k_0 < \infty$. Let ϖ be given, and let k_0 and l_0 be large integer parameters to be chosen later. In [5], $\varpi = 1/1168$, $k_0 = 3.5 \times 10^6$, and $l_0 = 180$, but one has the freedom to vary these parameters provided that a certain inequality (4) holds.

Let x be a large number, let $\mathcal{H}$ be an admissible tuple of cardinality k_0, and let $D, D_1, \mathcal{P}$ be as in the introduction. We introduce the Goldston-Pintz-Yildirim weight function [1]

$$\lambda(n) := \frac{1}{(k_0+l_0)!} \sum_{d | P(n); d \leq D} \mu(d) (\log \frac{D}{d})^{k_0+l_0}$$

where P is the polynomial $P(n) := \prod_{h \in \mathcal{H}}(n+h)$, and the sums

$$S_1 := \sum_{x \leq n \leq 2x} \lambda(n)^2$$

and

$$S_2 := \sum_{x \leq n \leq 2x} (\sum_{h \in \mathcal{H}} \theta(n+h)) \lambda(n)^2.$$

If one can show that

$$S_2 - (\log 3x) S_1 > 0 \tag{1}$$

for all sufficiently large x, then this implies an infinite number of translates $n + \mathcal{H}$ that contain at least two primes, giving $Q(k_0)$ as required.

To establish (1) we need upper bounds on S_1 and lower bounds on S_2. It turns out that one can establish bounds of the form

$$S_1 \leq \frac{1+\kappa_1}{(k_0+2l_0)!} \binom{2l_0}{l_0} \mathcal{G} x (\log D)^{k_0+2l_0} + o(x \log^{k_0+2l_0} x) \tag{2}$$

and

$$S_2 \geq \frac{k_0(1-\kappa_2)}{(k_0+2l_0+1)!} \binom{2l_0+2}{l_0+1} \mathcal{G} x (\log D)^{k_0+2l_0+1} + o(x \log^{k_0+2l_0+1} x) \tag{3}$$

where $\kappa_1, \kappa_2 > 0$ are two quantities depending on k_0, l_0, ϖ to be defined later, and $\mathcal{G}$ is the singular series

$$\mathcal{G} := \prod_p (1 - \frac{\nu_p}{p})(1 - \frac{1}{p})^{-k_0}$$

where $\nu_p = |C(p)|$ is the number of distinct residue classes occupied by $\mathcal{H}$ mod p. Except for the error terms κ_1, κ_2, these bounds are natural from sieve-theoretic considerations, and are unlikely to be improvable without new breakthroughs in sieve theory.

It is a standard fact that $0 < \mathcal{G} < \infty$. Since $\log x = \frac{4}{1+4\varpi} \log D$, we thus obtain (1) for sufficiently large x as soon as

$$\frac{k_0(1-\kappa_2)}{(k_0+2l_0+1)!}\binom{2l_0+2}{l_0+1} > \frac{4}{1+4\varpi}\frac{1+\kappa_1}{(k_0+2l_0)!}\binom{2l_0}{l_0}$$

which simplifies to

$$(1+4\varpi)(1-\kappa_2) > (1+\frac{1}{2l_0+1})(1+\frac{2l_0+1}{k_0})(1+\kappa_1) \tag{4}$$

If $\varpi > 0$, this inequality can be satisfied if κ_1, κ_2 are small enough and k_0, l_0 are large enough. Note that

$$(1+\frac{1}{2l_0+1})(1+\frac{2l_0+1}{k_0}) \geq (1+\frac{1}{k_0^{1/2}})^2$$

and so a necessary condition for (4) to be satisfied is that

$$k_0 > \frac{1}{((1+4\varpi)^{1/2}-1)^2}$$

which places a theoretical limit as to how small a value of k_0 one can extract from a given value of ϖ. In particular, with the choice $\varpi = 1/1168$ from Theorem 3, one cannot hope for a better value of k_0 than $341,640$.

This is an order of magnitude better than the value $k_0 = 3.5 \times 10^6$ in Theorem 2; this is due to the need to get good bounds on κ_1, κ_2. There is thus scope to improve k_0 a fair bit without hitting the full limits of the Goldston-Pintz-Yildirim method or without improving ϖ by improving the bounds on κ_1, κ_2.

In [5, §4] it is shown that one can take

$$\kappa_1 = \delta_1(1+\delta_2^2+k_0\log(1+\frac{1}{4\varpi}))\binom{k_0+2l_0}{k_0} \tag{5}$$

where

$$\delta_1 := (1+4\varpi)^{-k_0}$$

and δ_2 is any quantity for which one has the upper bound

$$\sum_{q|\mathcal{P}^*; q<D} \frac{\varrho_1(q)}{q} \leq \delta_2 + o(1) \tag{6}$$

where

$$\mathcal{P}^* := \prod_{D_1 \leq p < D} p$$

and $\varrho_1(q)$ is the multiplicative function on square-free integers with $\varrho_1(p) = \nu_p$ for all p; see [5, (4.15)]. Similarly, in [5, §5] it is shown that one can take

$$\kappa_2 = \delta_1(1+4\varpi)(1+\delta_2^2+\log(1+\frac{1}{4\varpi})k_0)\binom{k_0+2l_0+1}{k_0-1} \tag{7}$$

which simplifies to

$$\kappa_2 = (1+4\varpi)\kappa_1\frac{k_0(k_0+2l_0+1)}{(2l_0+1)(2l_0+2)}.$$

Now we turn to the problem of estimating δ_2. Zhang does this as follows. Firstly, we have $\nu_p \leq k_0$ for all primes p, so that $\varrho_1(q) \leq k_0^j$ when q is the product of exactly j primes. Thus one can bound the left-hand side of (6) by

$$\sum_{j=0}^\infty k_0^j \sum_{D_1 \leq p_1 < \ldots < p_j < D : p_1 \ldots p_j < D} \frac{1}{p_1 \ldots p_j}.$$

Note that $D = D_1^{1+\frac{1}{4\varpi}}$, so we can restrict j to $j \leq \frac{1}{4\varpi}$ (assuming that $\frac{1}{4\varpi}$ is an integer; note that it is equal to 292 in the case $\varpi = 1/1168$). If we then discard the $p_1 \ldots p_j < D$ constraint, we can then bound the above expression by

$$\sum_{j=0}^{1/4\varpi} \frac{k_0^j}{j!}(\sum_{D_1 \leq p < D} \frac{1}{p})^j.$$

By the prime number theorem we have

$$\sum_{D_1 \leq p < D} \frac{1}{p} = \log\log D - \log\log D_1 + o(1) = \log(1+\frac{1}{4\varpi}) + o(1)$$

and so Zhang obtains the value

$$\delta_2 := \sum_{j=0}^{1/4\varpi} \frac{(k_0 \log(1+\frac{1}{4\varpi}))^j}{j!} \tag{8}$$

for δ_2.

This is however a bit wasteful because we can take further advantage of the $p_1 \ldots p_j < D$ constraint by the method of Buchstab iteration. For any $x, y > 0$, we define the quantity

$$\Phi(x,y) := \sum_{j=0}^\infty k_0^j \sum_{y \leq p_1 < \ldots < p_j : p_1 \ldots p_j < x} \frac{1}{p_1 \ldots p_j} \tag{9}$$

then we can bound the left-hand side of (6) by $\Phi(D_1, D)$. We observe that

$$\Phi(x,y) = 1 \tag{10}$$

when $y > x$, while in general we have the *Buchstab identity*

$$\Phi(x,y) \leq 1 + k_0 \sum_{y \leq p < x} \frac{1}{p}\Phi(\frac{x}{p}, p) \tag{11}$$

as can be seen by isolating the smallest prime p_1 in all the terms in (9) with $j \geq 1$. (This inequality is very close to being an identity.) We can iterate this identity to obtain the following conclusion:

Lemma 4. *For any $n \geq 1$, we have*

$$\Phi(x, y) \leq \prod_{j=1}^{n-1} (1 + k_0 \log(1 + \frac{1}{j})) + o(1)$$

whenever y is large and $x \leq y^n$, with the error $o(1)$ going to zero as $y \to \infty$ uniformly in x for fixed n.

Proof. Write $A_n := \prod_{j=1}^{n-1}(1 + k_0 \log(1 + \frac{1}{j}))$. We prove the bound $\Phi(x, y) \leq A_n + o(1)$ by strong induction on n. The case $n = 1$ follows from (10). Now suppose that $n > 1$ and that the claim has already been proven for smaller n. Let $x \leq y^n$. Note that $\frac{x}{p} \leq p^j$ whenever $p \geq x^{\frac{1}{j+1}}$. We thus have from (11) and the induction hypothesis that

$$\Phi(x, y) \leq 1 + k_0 \sum_{j=1}^{n-1} \sum_{x^{\frac{1}{j+1}} \leq p < x^{\frac{1}{j}}} \frac{1}{p}(A_j + o(1));$$

applying the prime number theorem we have

$$\sum_{x^{\frac{1}{j+1}} \leq p < x^{\frac{1}{j}}} \frac{1}{p}(A_j + o(1)) = A_j \log(1 + \frac{1}{j}) + o(1)$$

and the claim follows from the telescoping identity

$$A_n = 1 + k_0 \sum_{j=1}^{n-1} A_j \log(1 + \frac{1}{j}).$$

□

From this lemma with $n := 1 + \frac{1}{4\varpi}$, $x = D$, and $y = D_1$ we see that we can take δ_2 to be

$$\delta_2 = \prod_{j=1}^{\frac{1}{4\varpi}} (1 + k_0 \log(1 + \frac{1}{j})). \tag{12}$$

This is roughly equal to $k_0^{1/4\varpi}/(\frac{1}{4\varpi} - 1)!$, which improves over (8) by a factor of about $(\log(1 + \frac{1}{4\varpi}))^{1/4\varpi}$.

3. Further improvement in κ_1, κ_2

The dominant terms in Zhang's formulae (5), (7) for κ_1, κ_2 come from the quantity δ_2 that bounds (6). Unfortunately this quantity is quite large, even if one replaces Zhang's bound (8) for δ_2 with the improved bound (12). But a further analysis of Zhang's argument allows one to replace the δ_2 terms in (5), (7) by significantly better behaved quantities.

The $\delta_1 \delta_2^2$ term in (5) comes through the estimate

$$|\Sigma_2| \leq \frac{\delta_1 \delta_2^2}{k_0!(l_0!)^2} \mathcal{G} (\log D)^{k_0+2l_0} + o(\log^{k_0+2l_0} x) \tag{13}$$

from [5, (4.15)], where Σ_2 is the quantity

$$\Sigma_2 := \sum_{d_0 \leq x^{1/4}, d_0 | \mathcal{P}} \sum_{d_1, d_2 | \mathcal{P}} \frac{\mu(d_1 d_2) \varrho_1(d_0 d_1 d_2)}{d_0 d_1 d_2} g(d_0 d_1) g(d_0 d_2)$$

where

$$g(y) := \frac{1}{(k_0+l_0)!}(\log \frac{D}{y})^{k_0+l_0}$$

if $y < D$ and $g(y) = 0$ with $y \geq D$. Similarly, the $\delta_1(1+4\varpi)\delta_2^2$ term in (7) comes through the estimate

$$|\Sigma_2'| \leq \frac{\delta_1(1+4\varpi)\delta_2^2}{(k_0-1)!((l_0+1)!)^2}\mathcal{G}(\log D)^{k_0+2l_0+1} + o(\log^{k_0+2l_0+1} x)$$

where Σ_2' is the quantity

$$\Sigma_2 := \sum_{d_0 \leq x^{1/4}, d_0|\mathcal{P}} \sum_{d_1,d_2|\mathcal{P}} \frac{\mu(d_1d_2)\varrho_2(d_0d_1d_2)}{\varphi(d_0d_1d_2)} g(d_0d_1)g(d_0d_2)$$

where φ is the Euler totient function, and ϱ_2 is defined similarly to ϱ_1 but with $\varrho_2(p)$ equal to $\nu_p - 1$ instead of ν_p.

We now obtain improved bounds on Σ_2 and Σ_2' that replaces $\delta_1\delta_2^2$ by a significantly smalelr quantity, leading to an improved value of κ_1 and κ_2. We begin by reviewing the proof of (13). From [5, (4.11)] one has

$$\Sigma_2 = \sum_{d < x^{1/4}D_0; d|\mathcal{P}} \frac{\varrho_1(d)\theta^*(d)}{d}\mathcal{A}_1^*(d)^2 + o(\log^{k_0+2l_0} x) \tag{14}$$

where D_0 is the relatively small quantity

$$D_0 := \exp(\log^{1/k_0} x),$$

$\theta^*(d)$ is the quantity

$$\theta^*(d) := \sum_{d_0q=d; d_0<x^{1/4}; q<D_0} \frac{\mu(q)\varrho_1(q)}{q},$$

and $\mathcal{A}_1^*(d)$ is the quantity

$$\mathcal{A}_1^*(d) := \sum_{r:(r,d)=1; r|\mathcal{P}} \frac{\mu(r)\varrho_1(r)g(dr)}{r}.$$

As observed just before [5, (4.12)], one can use Möbius inversion to write

$$\mathcal{A}_1^*(d) = \sum_{q|\mathcal{P}^*} \frac{\varrho_1(q)}{q}\mathcal{A}_1(dq)$$

where

$$\mathcal{P}^* := \prod_{D_1 \leq p < D} p$$

and

$$\mathcal{A}_1(d) := \sum_{r:(r,d)=1} \frac{\mu(r)\varrho_1(r)g(dr)}{r}.$$

Furthermore, in [5, Lemma 2] the bound

$$\mathcal{A}_1(d) = \frac{\theta_1(d)}{l_0!}\mathfrak{G}(\log \frac{D}{d})^{l_0} + o(\log^{l_0} x)$$

for $d \leq D$ is established, where

$$\theta_1(d) := \mu^2(d)\prod_{p|d}(1-\frac{\nu_p}{p})^{-1}.$$

Of course one has $\mathcal{A}_1(d) = 0$ for $d > D$. From the prime number theorem we also have

$$\sum_{q<D;q|\mathcal{P}^*} \frac{\varrho_1(d)}{d} = O(1)$$

(with the implied constant depending on k_0, ϖ, $\mathfrak{G}$, l_0). From this we see that

$$\mathcal{A}_1^*(d) = \sum_{q|\mathcal{P}^*:dq\leq D} \frac{\varrho_1(q)}{q} \frac{\theta_1(dq)}{l_0!} \mathfrak{G}(\log \frac{D}{dq})^{l_0} + o(\log^{l_0} x).$$

In [5, (4.12)], it is observed that

$$\theta_1(q) = 1 + O(D_1^{-1})$$

when $q|\mathcal{P}^*$ and $q < D$; also, $\log \frac{D}{dq} \leq \log D$, thus

$$|\mathcal{A}_1^*(d)| \leq \frac{\theta_1(d)}{l_0!} \mathfrak{G}(\log D)^{l_0} \sum_{q|\mathcal{P}^*:dq\leq D} \frac{\varrho_1(q)}{q} + o(\log^{l_0} x). \tag{15}$$

In [5, (4.13)], the condition $dq \leq D$ is replaced by $q \leq D$, and then by (6) one has

$$|\mathcal{A}_1^*(d)| \leq \frac{\delta_2}{l_0!} \mathfrak{G}\theta_1(d)(\log D)^{l_0} + o(\log^{l_0} x) \tag{16}$$

and thus[1] by (14)

$$|\Sigma_2| \leq \frac{\delta_2^2}{(l_0!)^2} \mathfrak{G}^2 \sum_{d<x^{1/4}D_0;d|\mathcal{P}} \frac{\varrho_1(d)\theta^*(d)\theta_1(d)^2}{d} (\log D)^{2l_0} + o(\log^{k_0+2l_0} x).$$

The contribution of those d with $x^{1/4} \leq d < x^{1/4}D_0$ turns out to be $(\log \frac{D}{d})^{l_0}$; see the discussion after [5, (4.7)]. In this regime we have

$$\theta^*(d) = \theta_1(d)^{-1} + O(\tau_{k_0+1}(d)D_0^{-1})$$

(see the equation after [5, (4.8)]), so

$$|\Sigma_2| \leq \frac{\delta_2^2}{(l_0!)^2} \mathfrak{G}^2 \sum_{d<x^{1/4};d|\mathcal{P}} \frac{\varrho_1(d)\theta_1(d)}{d} (\log D)^{2l_0} + o(\log^{k_0+2l_0} x).$$

But (a straightforward generalisation of) [5, Lemma 3] reveals that

$$\sum_{d<y} \frac{\varrho_1(d)\theta_1(d)}{d} = (\frac{1}{k_0!\mathfrak{G}} + o(1))(\log y)^{k_0} \tag{17}$$

for any $y > 1$; applying this with $y = x^{1/4}$, the bound (13) then follows.

Reviewing the proof of (13), we see that there are at least three possible inefficiencies:

(i) The replacement of the constraint $dq \leq D$ by $q \leq D$;
(ii) the application of (17) with $y = x^{1/4}$; and
(iii) the bounding of $\log \frac{D}{dq}$ by $\log D$.

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[1] Strictly speaking, to get the $o()$ error term here one has to tighten some of the $o()$ errors previously, in order to compensate for some losses of $\log\log x$ coming from the estimation of θ^*, but we will ignore this detail here.

The waste in step (iii) is fairly minor (it involves the exponent l_0 rather than k_0) so we will not attempt to recover it here. Instead we focus on the waste in steps (i) and (ii). The step (i) is wasteful when d is large, whereas step (ii) is wasteful when d is small, and so we can expect to get a significant improvement in all cases (especially since the exponent k_0 involved in both steps is large).

Consider the contribution to Σ_2 of those d for which $D_1^{-j-1} D \leq d < D_1^{-j} D$ for some $1 \leq j \leq \frac{1}{4\varpi}$. (This covers all regimes except the case $x^{1/4} \leq d < x^{1/4} D_0$, which we already know to be negligible.) In this regime the constraint $dq \leq D$ can be replaced by $q \leq D_1^{j+1}$ instead of $q \leq D$, which by Lemma 4 gives

$$\sum_{q|\mathcal{P}^*: dq \leq D} \frac{\varrho_1(q)}{q} \leq \delta_{2,j} + o(1)$$

where

$$\delta_{2,j} := \prod_{i=1}^{j} (1 + k_0 \log(1 + \frac{1}{i}))$$

and so by (15) we may improve (16) to

$$|\mathcal{A}_1^*(d)| \leq \frac{\delta_{2,j}}{l_0!} \mathfrak{G} \theta_1(d) (\log D)^{l_0} + o(\log^{l_0} x).$$

Repeating the previous arguments, we conclude that

$$|\Sigma_2| \leq \sum_{j=1}^{1/4\varpi} \frac{\delta_{2,j}^2}{(l_0!)^2} \mathfrak{G}^2 \sum_{D_1^{-j-1} D \leq d < D_1^{-j} D; d|\mathcal{P}} \frac{\varrho_1(d)\theta_1(d)}{d} (\log D)^{2l_0} + o(\log^{k_0+2l_0} x).$$

Applying (17) we conclude that

$$|\Sigma_2| \leq \sum_{j=1}^{1/4\varpi} \frac{\delta_{2,j}^2}{(k_0!(l_0!)^2} \mathfrak{G} (\log D_1^{-j} D)^{k_0} (\log D)^{2l_0} + o(\log^{k_0+2l_0} x).$$

which improves (13) to

$$|\Sigma_2| \leq \frac{\sum_{j=1}^{1/4\varpi} \delta_1^j \delta_{2,j}^2}{k_0!(l_0!)^2} \mathcal{G} (\log D)^{k_0+2l_0} + o(\log^{k_0+2l_0} x)$$

and thus improves κ_1 to

$$\kappa_1 = (\delta_1 + \sum_{j=1}^{1/4\varpi} \delta_1^j \delta_{2,j}^2 + \delta k_0 \log(1 + \frac{1}{4\varpi})) \binom{k_0 + 2l_0}{k_0}.$$

A similar argument improves κ_2 to

$$\kappa_2 = (\delta_1(1+4\varpi) + \sum_{j=1}^{1/4\varpi} \delta_1^j (1+4\varpi)^j \delta_{2,j}^2 + \delta(1+4\varpi) k_0 \log(1 + \frac{1}{4\varpi})) \binom{k_0 + 2l_0 + 1}{k_0 - 1}.$$

This appears to be a significant improvement over the previous values of κ_1, κ_2, as δ_1 is extremely small in practice, leading to the $j = 1$ terms in the above sum being dominant (and these terms are basically of the same order of magnitude as the other two terms, as opposed to being about δ_2 times larger).

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