STABILITY THEORY FOR NEUMANN EIGENFUNCTIONS

1. A Sobolev inequality

Lemma 1.1. *Let $ABCD$ be a parallelogram, and let u be a C^2 function on $ABCD$. Let α be the angle subtended by A. Then*

$$|u(A) - u(B) + u(C) - u(D)| \leqslant \frac{1}{\sin(\alpha)} \int_{ABCD} |\nabla^2 u|.$$

Proof. We may normalise $A = 0$, so that $C = B + D$. From two applications of the fundamental theorem of calculus one has

$$\int_0^1 \int_0^1 \partial_{st} u(sB + tD) = u(A) - u(B) + u(C) - u(D).$$

The left-hand side can be rewritten as

$$\frac{1}{|ABCD|} \int_{ABCD} (B \cdot \nabla)(D \cdot \nabla) u.$$

Since

$$|ABCD| = |B||D| \sin(\alpha)$$

the claim follows. □

Lemma 1.2. *Let ABC be a triangle with angles α, β, γ, and let u be a C^2 function on ABC that obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of ABC. Then for any $P \in ABC$, one has*

$$|u(P) - u(Q)| \leqslant \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.$$

Proof. Let $X \in ABC$. Let $D, E \in AB$, $F, G \in BC, H, I \in AC$ be the points such that $ADXI, BFXE, CHXG$ are parallelograms; thus D, G are the intersections with AB, BC respectively of the line through X parallel to AC, and so forth. Then from the preceding lemma one has

$$|u(X) + u(A) - u(D) - u(I)| \leqslant \frac{1}{\sin(\alpha)} \int_{ADXI} |\nabla^2 u|$$
$$|u(X) + u(B) - u(F) - u(E)| \leqslant \frac{1}{\sin(\beta)} \int_{BFXE} |\nabla^2 u|$$
$$|u(X) + u(C) - u(H) - u(G)| \leqslant \frac{1}{\sin(\gamma)} \int_{CHXG} |\nabla^2 u|.$$

Also, by reflecting the triangles DEX, FGX, XHI across the Neumann boundary and using the previous lemma, we see that

$$|u(X)+u(X)-u(D)-u(E)| \leqslant \frac{2}{\sin(\gamma)}\int_{DEX}|\nabla^2 u|$$
$$|u(X)+u(X)-u(F)-u(G)| \leqslant \frac{2}{\sin(\alpha)}\int_{FGX}|\nabla^2 u|$$
$$|u(X)+u(X)-u(H)-u(I)| \leqslant \frac{2}{\sin(\beta)}\int_{XHI}|\nabla^2 u|.$$

Summing the latter three combinations of u and subtracting the former three using the triangle inequality, we conclude that

$$|3u(X)-u(A)-u(B)-u(C)| \leqslant \frac{2}{\min(\sin(\alpha),\sin(\beta),\sin(\gamma))}\int_{ABC}|\nabla^2 u|.$$

Setting $X = P, Q$ and subtracting, we obtain the claim. □

Corollary 1.3. *Let ABC be a triangle with angles α, β, γ, and let u be a C^2 function on ABC which is smooth up to the boundary except possibly at the vertices A, B, C, and which obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of ABC, and has mean zero on ABC. Then*

$$\|u\|_{L^\infty(ABC)} \leqslant \frac{4}{3\min(\sin(\alpha),\sin(\beta),\sin(\gamma))}|ABC|^{1/2}\|\Delta u\|_{L^2(ABC)}.$$

Proof. If u has mean zero, then $\|u\|_{L^\infty(ABC)}$ is bounded by $|u(P)-u(Q)|$ for some $P, Q \in ABC$. From the previous lemma we thus have

$$\|u\|_{L^\infty(ABC)} \leqslant \frac{4}{3\min(\sin(\alpha),\sin(\beta),\sin(\gamma))}|ABC|^{1/2}\|\nabla^2 u\|_{L^2(ABC)}.$$

It will thus suffice to show the Bochner-Weitzenbock identity

$$\int_{ABC}|\nabla^2 u|^2 = \int_{ABC}|\Delta u|^2.$$

But this can be accomplished by two integration by parts, using the smoothness and Neumann boundary hypotheses on u (and a regularisation argument if necessary to cut away from the vertices) **more details needed here**. □

2. Schwarz-Christoffel

Let $0 < \alpha, \beta, \gamma < \pi$ be angles adding up to π, then we can define a Schwarz-Christoffel map $\Phi_{\alpha,\beta} : \mathbb{H} \to ABC$ from the half-plane $\mathbb{H} := \{z : \Im(z) > 0\}$ to a triangle ABC with angles α, β, γ by the formula

$$\Phi_{\alpha,\beta}(z) := \int_0^z \frac{d\zeta}{\zeta^{1-\alpha/\pi}(1-\zeta)^{1-\beta/\pi}},$$

where the integral is over any contour from 0 to z in $\mathbb{H}$, and one chooses the branch cut to make both factors in the denominator positive real on the interval $[0,1]$. Thus the vertices of the triangle are given by

$$\begin{aligned}
A &:= \Phi_{\alpha,\beta}(0) = 0 \\
B &:= \Phi_{\alpha,\beta}(1) = \int_0^1 \frac{dt}{t^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} = \frac{\Gamma(\alpha/\pi)\Gamma(\beta/\pi)}{\Gamma((\alpha+\beta)/\pi)} \\
C &:= \Phi_{\alpha,\beta}(\infty) = -e^{i\alpha}\int_0^{-\infty} \frac{dt}{|t|^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} \\
&= e^{i\alpha}\int_1^{\infty} \frac{ds}{(s-1)^{1-\alpha/\pi}s^{1-\beta/\pi}} \\
&= e^{i\alpha}\int_0^{1} \frac{dv}{(v-1)^{1-\alpha/\pi}v^{1-\gamma/\pi}} \\
&= e^{i\alpha}\frac{\Gamma(\alpha/\pi)\Gamma(\gamma/\pi)}{\Gamma((\alpha+\gamma)/\pi)}
\end{aligned}$$

where we have used the beta function identity

$$\int_0^1 \frac{dt}{t^{1-x}(1-t)^{1-y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and the changes of variable $s = 1-t$, $v = 1/s$. In particular, the area of the triangle ABC can be expressed as

$$|ABC| = \frac{1}{2}|B||C|\sin(\alpha) = \frac{\Gamma(\alpha/\pi)^2\Gamma(\beta/\pi)\Gamma(\gamma/\pi)}{2\Gamma((\alpha+\beta)/\pi)\Gamma((\alpha+\gamma)/\pi)}\sin(\alpha)$$

which can be simplified using the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ as the more symmetric expression

$$|ABC| = \frac{1}{2\pi^2}\Gamma(\alpha/\pi)^2\Gamma(\beta/\pi)^2\Gamma(\gamma/\pi)^2\sin(\alpha)\sin(\beta)\sin(\gamma). \tag{2.1}$$

We write

$$|\Phi'_{\alpha,\beta}(z)| = e^{\omega(z)}$$

where $\omega = \omega_{\alpha,\beta}$ is the harmonic function

$$\omega(z) := (\frac{\alpha}{\pi} - 1)\log|z| + (\frac{\beta}{\pi} - 1)\log|1-z|. \tag{2.2}$$

If $u : ABC \to \mathbb{R}$ is a smooth function, and $\tilde{u} : \mathbb{H} \to \mathbb{R}$ is its pullback to the half-plane $\mathbb{H}$ defined by

$$\tilde{u} := u \circ \Phi_{\alpha,\beta}$$

then we have

$$\int_{ABC} u = \int_{\mathbb{H}} e^{2\omega}\tilde{u}.$$

In a similar vein we have the conformal invariance of the two-dimensional Dirichlet energy

$$\int_{ABC} |\nabla u|^2 = \int_{\mathbb{H}} |\nabla \tilde{u}|^2$$

and the conformal transformation of the Laplacian:

$$\Delta \tilde{u}(z) = e^{2\omega} \widetilde{\Delta u}.$$

In particular, the Rayleigh quotient

$$\int_{ABC} |\nabla u|^2 / \int_{ABC} |u|^2$$

with mean zero condition $\int_{ABC} u = 0$ becomes, when pulled back to $\mathbb{H}$, the Rayleigh quotient

$$\int_{\mathbb{H}} |\nabla \tilde{u}|^2 / \int_{\mathbb{H}} e^{2\omega} |\tilde{u}|^2$$

with mean zero condition $\int_{\mathbb{H}} e^{2\omega} \tilde{u} = 0$.

Let $u_2, u_3, \dots$ be an L^2-normalised eigenbasis for the Neumann Laplacian $-\Delta$ on ABC with eigenvalues $\lambda_2 \leqslant \lambda_3 \leqslant \dots$, thus

$$-\Delta u_k = \lambda_k u_k$$

on ABC with Neumann boundary data

$$n \cdot \nabla u_k = 0$$

and orthonormality

$$\int_{ABC} u_j u_k = \delta_{jk}$$

and mean zero condition

$$\int_{ABC} u_j = 0.$$

One can show that when ABC is acute-angled, these eigenfunctions are smooth except possibly at the vertices A, B, C, and are uniformly C^2. **add details here**

Pulling all this back to $\mathbb{H}$, we obtain transformed eigenfunctions $\tilde{u}_2, \tilde{u}_3, \dots$ on $\mathbb{H}$ to the conformal eigenfunction equation

$$-\Delta \tilde{u}_k = \lambda_k e^{2\omega} \tilde{u}_k \tag{2.3}$$

on $\mathbb{H}$ with Neumann boundary data

$$n \cdot \nabla \tilde{u}_k = 0 \tag{2.4}$$

and orthonormality

$$\int_{\mathbb{H}} e^{2\omega} \tilde{u}_j \tilde{u}_k = \delta_{jk} \tag{2.5}$$

and mean zero condition

$$\int_{\mathbb{H}} e^{2\omega} \tilde{u}_j = 0. \tag{2.6}$$

Now suppose that we vary the angle parameters α, β, γ smoothly with respect to some time parameter t, thus also varying the triangles ABC, eigenfunctions u_k and transformed eigenfunctions $\tilde{u}_k$, eigenvalues λ_k, and conformal factor ω. We will use dots to indicate time differentiation, thus for instance $\dot{\alpha} = \frac{d}{dt}\alpha$. Let us formally suppose that all of the above data vary smoothly (or at least C^1) in time **we will eventually need to justify this, of course**. Since $\alpha + \beta + \gamma = \pi$, we have

$$\dot{\alpha} + \dot{\beta} + \dot{\gamma} = 0.$$

The variation $\dot{\omega}$ of the conformal factor is explicitly computable from (2.2) as being a logarithmic weight:

$$\dot{\omega} = \frac{\dot{\alpha}}{\pi} \log |z| + \frac{\dot{\beta}}{\pi} \log |1 - z|.$$

Next, by (formally) differentiating (2.3) we obtain an equation for the variation $\dot{\tilde{u}}_k$ of the k^{th} eigenfunction:

$$-\Delta \dot{\tilde{u}}_k = \dot{\lambda}_k e^{2\omega} \tilde{u}_k + 2\lambda_k \dot{\omega} e^{2\omega} \tilde{u}_k + \lambda_k e^{2\omega} \dot{\tilde{u}}_k. \tag{2.7}$$

To solve this equation for $\dot{\tilde{u}}_k$, we observe from differentiating (2.4), (2.5), (2.6) that

$$\int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k = 0 \tag{2.8}$$

and

$$\int_{\mathbb{H}} e^{2\omega} \tilde{u}_k \dot{\tilde{u}}_k = 0$$

and

$$n \cdot \nabla \dot{\tilde{u}}_k = 0.$$

By eigenfunction expansion, we thus have

$$\dot{\tilde{u}}_k = \sum_{l \neq k} \left(\int_{\mathbb{H}} e^{2\omega} \tilde{u}_l \dot{\tilde{u}}_k \right) \tilde{u}_l \tag{2.9}$$

in a suitable sense (L^2 with weight e^ω). Now we evaluate the expression in parentheses. Integrating (2.7) against $\tilde{u}_l$ and using (2.5) reveals that

$$-\int_{\mathbb{H}} \Delta \dot{\tilde{u}}_k \tilde{u}_l = 2\lambda_k \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l + \lambda_k \int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l. \tag{2.10}$$

By Green's theorem and the Neumann conditions on $\dot{\tilde{u}}_k$ and $\tilde{u}_l$, the left-hand side is

$$-\int_{\mathbb{H}} \dot{\tilde{u}}_k \Delta \tilde{u}_l$$

which by (2.3) is equal to

$$\lambda_l \int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l.$$

Inserting this into (2.10) we see that

$$\int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l = \frac{2\lambda_k}{\lambda_l - \lambda_k} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l$$

and thus by (2.9)

$$\dot{\tilde{u}}_k = \sum_{l\neq k}(\frac{2\lambda_k}{\lambda_l-\lambda_k}\int_{\mathbb{H}}\dot{\omega}e^{2\omega}\tilde{u}_k\tilde{u}_l)\tilde{u}_l. \tag{2.11}$$

We can take Laplacians and conclude that

$$-\Delta\dot{\tilde{u}}_k = e^{2\omega}\sum_{l\neq k}(\frac{2\lambda_k\lambda_l}{\lambda_l-\lambda_k}\int_{\mathbb{H}}\dot{\omega}e^{2\omega}\tilde{u}_k\tilde{u}_l)\tilde{u}_l.$$

Set $k=2$, then $\frac{2\lambda_k\lambda_l}{\lambda_l-\lambda_k}$ is bounded in magnitude by $\frac{2\lambda_2\lambda_3}{\lambda_3-\lambda_2}$. From the orthonormality (2.5) and the Bessel inequality, we conclude that

$$(\int_{\mathbb{H}} e^{-2\omega}|\Delta\dot{\tilde{u}}_2|^2)^{1/2} \leqslant \frac{2\lambda_2\lambda_3}{\lambda_3-\lambda_2}(\int_{\mathbb{H}}|\dot{\omega}|^2e^{2\omega}\tilde{u}_2^2)^{1/2}. \tag{2.12}$$

If we change coordinates by writing

$$\dot{\tilde{u}} = \dot{u}\circ\Phi$$

we conclude that

$$(\int_{ABC}|\Delta\dot{u}_2|^2)^{1/2} \leqslant \frac{2\lambda_2\lambda_3}{\lambda_3-\lambda_2}(\int_{\mathbb{H}}|\dot{\omega}|^2e^{2\omega}\tilde{u}_2^2)^{1/2}.$$

Also, $\dot{u}_2$ has mean zero on ABC by (2.8). We conclude from Corollary 1.3 that

$$\|\dot{u}_2\|_{L^\infty} \leqslant \frac{4}{3\min(\sin(\alpha),\sin(\beta),\sin(\gamma))}|ABC|^{1/2}\frac{2\lambda_2\lambda_3}{\lambda_3-\lambda_2}(\int_{\mathbb{H}}|\dot{\omega}|^2e^{2\omega}\tilde{u}_2^2)^{1/2}.$$

Pulling back to $\mathbb{H}$, and estimating $\tilde{u}_2$ in L^∞ norm, we conclude that

$$\|\dot{\tilde{u}}_2\|_{L^\infty(\mathbb{H})} \leqslant X\|\tilde{u}_2\|_{L^\infty(\mathbb{H})}$$

where X is the explicit (but somewhat messy) quantity

$$X := \frac{4}{3\min(\sin(\alpha),\sin(\beta),\sin(\gamma))}|ABC|^{1/2}\frac{2\lambda_2\lambda_3}{\lambda_3-\lambda_2}(\int_{\mathbb{H}}|\dot{\omega}|^2e^{2\omega})^{1/2}.$$

This gives stability of the second eigenfunction in L^∞ norm, as long as there is an eigenvalue gap $\lambda_3-\lambda_2>0$.

One can compute one factor in the quantity X as follows. Observe that

$$\int_{\mathbb{H}} e^{2\omega} = |ABC| = \frac{1}{2\pi^2}\Gamma(\alpha/\pi)^2\Gamma(\beta/\pi)^2\Gamma(\gamma/\pi)^2\sin(\alpha)\sin(\beta)\sin(\gamma).$$

If we view α,β,γ (and hence ω) as varying linearly in time, and differentiate the above equation under the integral sign twice in time, we conclude that

$$\begin{aligned}4\int_{\mathbb{H}}|\dot{\omega}|^2e^{2\omega} &= \frac{d^2}{dt^2}\int_{\mathbb{H}}e^{2\omega}\\ &= (\dot{\alpha}^2\frac{\partial^2}{\partial\alpha^2}+2\dot{\alpha}\dot{\beta}\frac{\partial^2}{\partial\alpha\partial\beta}+2\dot{\alpha}\dot{\gamma}\frac{\partial^2}{\partial\alpha\partial\gamma}\\ &\quad+\dot{\beta}^2\frac{\partial^2}{\partial\beta^2}+2\dot{\beta}\dot{\gamma}\frac{\partial^2}{\partial\beta\gamma}+\dot{\gamma}^2\frac{\partial^2}{\partial\gamma^2})\\ &\quad(\frac{1}{2\pi^2}\Gamma(\alpha/\pi)^2\Gamma(\beta/\pi)^2\Gamma(\gamma/\pi)^2\sin(\alpha)\sin(\beta)\sin(\gamma)).\end{aligned}$$

This, in principle, expresses the factor $(\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega})^{1/2}$ in X as an explicit combination of trigonometric functions, gamma functions, and the first two derivatives of the gamma function. However, this formula is somewhat messy, to say the least.

3. An explicit example

Suppose we take the isosceles right-angled triangle

$$\alpha = \pi/2; \beta = \gamma = \pi/4$$

and move along the space of right-angled triangles by taking

$$\dot{\alpha} = 0; \dot{\beta} = 1; \dot{\gamma} = -1.$$

In this example we have

$$\frac{4}{3\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} = \frac{4\sqrt{2}}{3} \approx 1.8856$$

and

$$|ABC|^{1/2} = \frac{1}{2\pi}\Gamma(1/2)\Gamma(1/4)^2 \approx 3.70815$$

To put it another way, the sidelength $AB = AC$ is given by

$$|AB| = |AC| = \frac{1}{\sqrt{2}\pi}\Gamma(1/2)\Gamma(1/4)^2 \approx 5.24412.$$

We have

$$\lambda_2 = \frac{\pi^2}{|AB|^2}$$

and

$$\lambda_3 = \frac{2\pi^2}{|AB|^2}$$

and so

$$\frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} = \frac{4\pi^2}{|AB|^2} = 7.52814.$$

MAPLE tells me that

$$\frac{d^2}{dt^2}|ABC| \approx 40.836$$

and so

$$\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega} \approx 10.209$$

and so

$$X \approx 22.34.$$

4. Justifying the differentiability

In the above analysis, we assumed without proof that the eigenfunction u and eigenvalue λ behaved in a C^1 fashion with respect to smooth deformation of the triangle. We now justify this assertion.

It is convenient to first work with an affine model rather than a conformal one, using a fixed reference triangle Ω_0 instead of the half-plane $\mathbb{H}$ as the reference domain, and affine maps instead of Schwarz-Christoffel maps as the transformation maps. This is in order to keep the transformation maps smooth at the vertices; it comes at the cost of making the Neumann condition inhomogeneous. After we establish smoothness for this model, we will then change coordinates to the conformal model.

Namely, suppose one has a smooth family of triangles $ABC = ABC(t)$ with vertices $A(t), B(t), C(t)$ depending smoothly on a time parameter t (as is the case in the preceding discussion). We isolate the time zero triangle $\Omega_0 := ABC(0)$ as the reference domain, and view all the other triangles as affine images $\Omega(t) = F(t)(\Omega_0)$ of the reference triangle for some affine transformations $F(t) : \mathbb{R}^2 \to \mathbb{R}^2$ depending smoothly on t. Actually, we may normalise $A(t) = 0$ (say), so that the $F(t)$ are linear instead of affine.

Pulling the Rayleigh quotient from $\Omega(t)$ back to Ω_0, we see that the second eigenvalue $\lambda_2(t)$ comes from minimising the functional

$$\int_{\Omega_0} |F^{-1}(t)\nabla u|^2 / \int_{\Omega_0} |u|^2$$

among functions $u \in L^2(\Omega_0)$ of mean zero. Let us write $u_2(t)$ for a minimiser of this functional with unit norm, thus

$$\int_{\Omega_0} |u_2(t)|^2 = 1$$

and

$$\int_{\Omega_0} |F^{-1}(t)\nabla u_2(t)|^2 = \lambda_2(t);$$

as long as the second eigenvalue is simple, this uniquely determines $u_2(t)$ up to sign. The function u_2 is of course the second Neumann eigenfunction of $\Omega(t)$, pulled back to Ω_0.

We now compare $u_2(t)$ with $u_2(0)$ for t small, assuming an eigenvalue gap $\lambda_2(0) < \lambda_3(0)$. Let us write

$$u_2(t) = \cos\theta(t) u_2(0) + \sin\theta(t) v(t)$$

for some angle $\theta(t)$ and some $v(t)$ orthogonal to both $u_2(0)$ and 1, and of unit L^2 norm. By reflection we may also assume that $|\theta(t)| \leqslant \pi/2$. Because $u_2(t)$ achieves the minimum of the Rayleigh quotient, we see that

$$\int_{\Omega_0} |F^{-1}\nabla u_2(t)|^2 \leqslant \int_{\Omega_0} |F^{-1}\nabla u_2(0)|^2.$$

The left-hand side can be expanded as

$$\cos^2\theta(t)\int_{\Omega_0}|F^{-1}(t)\nabla u_2(0)|^2+2\cos\theta(t)\sin\theta(t)\int_{\Omega_0}(F^{-1}(t))^*F^{-1}(t)\nabla u_2(0)\cdot\nabla v(t)+\sin^2\theta(t)\int_{\Omega_0}|F^{-1}(t)$$

and thus, either $\theta(t)=0$ or

$$2\cos\theta(t)\int_{\Omega_0}(F^{-1}(t))^*F^{-1}(t)\nabla u_2(0)\cdot\nabla v(t)+\sin\theta(t)\int_{\Omega_0}|F^{-1}(t)\nabla v(0)|^2\leqslant\sin\theta(t)\int_{\Omega_0}|F^{-1}(t)\nabla u_2(0)|^2.$$

Note that as F depends smoothly on t, one has

$$\int_{\Omega_0}|F^{-1}(t)\nabla u_2(0)|^2=\lambda_2(0)+O(|t|)$$

and similarly

$$\int_{\Omega_0}|F^{-1}(t)\nabla v(0)|^2\geqslant(1-O(|t|))\int_{\Omega_0}|\nabla v(0)|^2\geqslant(1-O(|t|))\lambda_3(0)$$

thanks to the spectral theorem and the normalisation of v; here the implied constants are allowed to depend on everything except t. Thus, for sufficiently small t, we conclude that

$$|\sin\theta(t)|\int_{\Omega_0}|F^{-1}(t)\nabla v(0)|^2\ll|\int_{\Omega_0}(F^{-1}(t))^*F^{-1}(t)\nabla u_2(0)\cdot\nabla v(t)|.$$

The eigenfunction $u_2(0)$ obeys the eigenfunction equation $-\Delta u_2(0)=\lambda_2(0)u_2(0)$ with Neumann boundary condition $n\cdot\nabla u_2(0)=0$, and is bounded in H^2. In particular, by integration by parts

$$\int_{\Omega_0}\nabla u_2(0)\cdot\nabla v(t)=\int_{\Omega_0}\lambda_2(0)u_2(0)v(t)=0$$

and so

$$\int_{\Omega_0}(F^{-1}(t))^*F^{-1}(t)\nabla u_2(0)\cdot\nabla v(t)=O(|t|\int_{\Omega_0}|\nabla u_2(0)||\nabla v(t)|)$$

which by Cauchy-Schwarz gives

$$\int_{\Omega_0}(F^{-1}(t))^*F^{-1}(t)\nabla u_2(0)\cdot\nabla v(t)=O(|t|(\int_{\Omega_0}|F^{-1}(t)\nabla v(0)|^2)^{1/2}).$$

We thus have

$$\sin\theta(t)(\int_{\Omega_0}|F^{-1}(t)\nabla v(0)|^2)^{1/2}=O(|t|)$$

and thus $\theta(t)=O(|t|)$ and

$$\|\sin\theta(t)v\|_{H^1(\Omega_0)}=O(|t|).$$

In particular, we have the Lipschitz bound

$$\|u_2(t)-u_2(0)\|_{H^1(\Omega_0)}=O(|t|) \tag{4.1}$$

for t small enough. Comparing Rayleigh quotients then gives

$$|\lambda_2(t)-\lambda_2(0)|=O(|t|)$$

again for t small enough.

Now we move back to the conformal picture. Taking into account the difference between the affine coordinate transformations and the Schwarz-Christoffel transformations, the

H^1 bound (4.1) implies that the eigenfunctions $\tilde{u}_2(t)$, $\tilde{u}_2(0)$ obey an L^2 estimate of the form

$$(\int_{\mathbb{H}} |\tilde{u}_2(t,z) - \tilde{u}_2(0,z)|^2 e^{2\omega(0)})^{1/2} = O(|t|).$$

This can be established by using uniform C^0 bounds on $u_2(t)$ near vertices to handle the regions within $O(|t|^K)$ of the vertices A, B, C for some large constant K, and using (4.1) and a smooth deformation in t to control the remainder. If we write $\tilde{u}_2(t) = \tilde{u}_2(0) + t\tilde{v}(t)$, we thus have

$$\|\tilde{v}(t)e^{\omega(0)}\|_{L^2(\mathbb{H})} = O(1)$$

with $\tilde{v}(t)$ obeying Neumann boundary conditions. The same argument also gives the variant bounds

$$\|\tilde{v}(t)(e^{\omega(t)} - e^{\omega(0)})\|_{L^2(\mathbb{H})} = o(1).$$

Also, from the eigenfunction equation

$$-\Delta\tilde{u}_2(t) = -\lambda_2(t)e^{2\omega(t)}\tilde{u}_2(t)$$

and writing $\lambda_2(t) = \lambda_2(0) + t\gamma(t)$, $\omega(t) = \omega(0) + t\sigma(t)$, we see that

$$-\Delta\tilde{v}(t) = -e^{2\omega_0}((\lambda_2(0)+t\gamma(t))\frac{e^{2t\sigma(t)} - 1}{t}(\tilde{u}_2(0)+t\tilde{v}(t))+\gamma(t)\tilde{u}_2(0)+\lambda_2(0)\tilde{v}(t)+t\gamma(t)\tilde{v}(t)).$$

Since $\gamma(t) = O(1)$, the previous bounds on $\tilde{v}$ give

$$\|e^{-\omega_0}\Delta\tilde{v}\|_{L^2(\mathbb{H})} = O(1)$$

and in particular (on integrating this against $\tilde{v}(t)e^{\omega(0)}$)

$$\|\nabla\tilde{v}\|_{L^2(\mathbb{H})} = O(1).$$

From this and Hardy's inequality one obtains (after some calculation) that

$$(-\Delta - \lambda_2 e^{2\omega_0})\tilde{v}(t) = -e^{2\omega_0}(2\lambda_2(0)\sigma(0)\tilde{u}_2(0) + \gamma(t)\tilde{u}_2(0) + e(t))$$

where $\|e(t)\|_{L^2(\mathbb{H})} = o(1)$. Integrating this against $\tilde{u}_2(0)$, we obtain that

$$0 = -2\lambda_2(0)\sigma(0)\int_{\mathbb{H}} e^{2\omega_0}\sigma(0)\tilde{u}_2(0)^2 + \gamma(t) + o(1)$$

which among other things shows that $\gamma(t)$ is continuous at $t = 0$, and we now have

$$(-\Delta - \lambda_2 e^{2\omega_0})\tilde{v}(t) = -e^{2\omega_0}(2\lambda_2(0)\sigma(0)\tilde{u}_2(0) + \gamma(0)\tilde{u}_2(0) + e'(t))$$

where $\|e(t)\|_{L^2(\mathbb{H})} = o(1)$. Solving this inhomogeneous eigenfunction equation, we see that the component of $\tilde{v}(t)$ orthogonal to $\tilde{u}_2(0)$ in $L^2(\mathbb{H}, e^{2\omega_0})$ is continuous at $t = 0$ in the $L^2(\mathbb{H}, e^{2\omega_0})$ norm. As for the component parallel to $\tilde{u}_2(0)$, we use the normalisation

$$\int_{\mathbb{H}} |\tilde{u}_2(t)|^2 e^{2\omega(t)} = \int_{\mathbb{H}} |\tilde{u}_2(0)|^2 e^{2\omega_0} = 1$$

which we rewrite using $\tilde{u}_2(t) = \tilde{u}_2(0) + t\tilde{v}(t)$ as

$$2\int_{\mathbb{H}} \tilde{u}_2(0)\tilde{v}(0)e^{2\omega_0} = \int_{\mathbb{H}} |\tilde{u}_2(0)|^2\frac{e^{2\omega_0} - e^{2\omega(t)}}{t} + 2\int_{\mathbb{H}} \tilde{u}_2(0)\tilde{v}(0)(e^{2\omega_0} - e^{2\omega(t)}) + t\int_{\mathbb{H}} |\tilde{v}(0)|^2 e^{2\omega(t)}.$$

The terms in the right-hand side can be evaluated to be $-2\int_{\mathbb{H}} |\tilde{u}_2(0)|^2\omega_0 + o(1)$ (for the first integral on the right-hand side we have to treat the region very close to the vertices using C^0 bounds on $\tilde{u}_2$). Thus we see that this component also depends continuously on t. This gives differentiability of $\tilde{u}_2$ and λ_2 in t (and even gives the correct explicit formula for the derivative).

5. Combining the second and third eigenfunction

The equation (2.11) describes the evolution of eigenfunctions such as $\tilde{u}_2$ and $\tilde{u}_3$. Unfortunately these equations contain a term that has a $\lambda_2 - \lambda_3$ in the denominator, and thus look useless in the case that λ_2 and λ_3 come close to each other. However, we can eliminate this term by considering the evolution of $\tilde{u}_2$ and $\tilde{u}_3$ jointly, by working with the circle $\{\cos(\theta)\tilde{u}_2 + \sin(\theta)\tilde{u}_3 : \theta \in \mathbb{R}/2\pi\mathbb{Z}\}$.

Let's see how. From (2.11) we have

$$\dot{\tilde{u}}_2 = a\tilde{u}_3 + f_2$$
$$\dot{\tilde{u}}_3 = -a\tilde{u}_2 + f_3$$

where

$$a := \frac{2\lambda_2}{\lambda_3 - \lambda_2}(\int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_2 \tilde{u}_3)$$
$$f_2 := \sum_{l>3} (\frac{2\lambda_2}{\lambda_l - \lambda_2} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_2 \tilde{u}_l) \tilde{u}_l$$
$$f_3 := \sum_{l>3} (\frac{2\lambda_3}{\lambda_l - \lambda_3} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_3 \tilde{u}_l) \tilde{u}_l - 2(\int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_2 \tilde{u}_3) \tilde{u}_2.$$

If we let $\theta : \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ be a smooth function, we thus see that

$$\frac{d}{dt}(\cos(\theta)\tilde{u}_2 + \sin(\theta)\tilde{u}_3) = (a - \dot{\theta})(-\sin(\theta)\tilde{u}_2 + \cos(\theta)\tilde{u}_3) + \cos(\theta) f_2 + \sin(\theta) f_3.$$

Suppose we select θ so that

$$\dot{\theta} = a \tag{5.1}$$

then we conclude that

$$\frac{d}{dt}(\cos(\theta)\tilde{u}_2 + \sin(\theta)\tilde{u}_3) = \cos(\theta) f_2 + \sin(\theta) f_3.$$

Let us compute the sup norm

$$\|\frac{d}{dt}(\cos(\theta)\tilde{u}_2 + \sin(\theta)\tilde{u}_3)\|_{L^\infty(\mathbb{H})}.$$

From Corollary 1.3 (transforming between $\mathbb{H}$ and ABC) we may bound this by

$$\frac{4}{3\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} (\int_{\mathbb{H}} e^{-2\omega} (|\cos(\theta)\Delta f_2 + \sin(\theta)\Delta f_3|^2)^{1/2}.$$

Note that

$$\Delta f_2 := \sum_{l>3} (\frac{2\lambda_l \lambda_2}{\lambda_l - \lambda_2} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_2 \tilde{u}_l) \tilde{u}_l$$
$$\Delta f_3 := \sum_{l>3} (\frac{2\lambda_l \lambda_3}{\lambda_l - \lambda_3} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_3 \tilde{u}_l) \tilde{u}_l - 2\lambda_2(\int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_2 \tilde{u}_3) \tilde{u}_2.$$

Note that the quantities $\frac{2\lambda_l\lambda_2}{\lambda_l-\lambda_2}$, $\frac{2\lambda_l\lambda_3}{\lambda_l-\lambda_3}$, $2\lambda_2$ are all bounded by $\frac{2\lambda_3\lambda_4}{\lambda_4-\lambda_3}$. Arguing as in the proof of (2.12), we conclude that

$$(\int_{\mathbb{H}} e^{-2\omega}|\Delta f_2|^2)^{1/2} \leqslant \frac{2\lambda_3\lambda_4}{\lambda_4-\lambda_3}(\int_{\mathbb{H}} |\dot\omega|^2 e^{2\omega}\tilde u_2^2)^{1/2}$$

and

$$(\int_{\mathbb{H}} e^{-2\omega}|\Delta f_3|^2)^{1/2} \leqslant \frac{2\lambda_3\lambda_4}{\lambda_4-\lambda_3}(\int_{\mathbb{H}} |\dot\omega|^2 e^{2\omega}\tilde u_3^2)^{1/2}.$$

Estimating $\tilde u_3, \tilde u_2$ in L^∞ norm, and also estimating $|\cos(\theta)| + |\sin(\theta)|$ crudely[1] by $\sqrt{2}$, we conclude that

$$\sup_\theta \|\frac{d}{dt}(\cos(\theta)\tilde u_2 + \sin(\theta)\tilde u_3)\|_{L^\infty(\mathbb{H})} \leqslant X' \sup_\theta \|\cos(\theta)\tilde u_2 + \sin(\theta)\tilde u_3\|_{L^\infty(\mathbb{H})}$$

where

$$X' := \sqrt{2}\frac{4}{3\min(\sin(\alpha),\sin(\beta),\sin(\gamma))}|ABC|^{1/2}\frac{2\lambda_3\lambda_4}{\lambda_4-\lambda_3}(\int_{\mathbb{H}} |\dot\omega|^2 e^{2\omega})^{1/2}.$$

Combining this with Gronwall's inequality, we conclude:

Theorem 5.1 (Stability of $\tilde u_2, \tilde u_3$)**.** *Let $[t_1, t_2]$ be a time interval. Then for any $\theta_2 \in \mathbb{R}/2\pi\mathbb{Z}$, there exists $\theta_1 \in \mathbb{R}/2\pi\mathbb{Z}$ such that*

$$\|(\cos(\theta_2)\tilde u_2(t_2)+\sin(\theta_2)\tilde u_3(t_2))-(\cos(\theta_1)\tilde u_2(t_1)+\sin(\theta_1)\tilde u_3(t_1))\|_{L^\infty(\mathbb{H})} \leqslant \exp(\int_{t_1}^{t_2} X'(t)\ dt)\sup_\theta \|\cos(\theta)\tilde u_2$$

In particular, if ABC is equal to the equilateral triangle at time t_1 (for which $\lambda_2 = \lambda_3$), then setting $\theta_2 = 0$, we see that

$$\|\tilde u_2(t_2) - \tilde u_2(t_1)\|_{L^\infty(\mathbb{H})} \leqslant \exp(\int_{t_1}^{t_2} X'(t)\ dt)\|\tilde u_2'(t_1)\|_{L^\infty(\mathbb{H})}$$

for *some* L^2-normalised second eigenfunctions $\tilde u_2(t_1), \tilde u_2'(t_1)$.

6. Perturbing from the equilateral triangle

Let ABC be an equilateral triangle. Suppose we have a function u which is close in L^∞ norm to an L^2-normalised second eigenfunction u_2 in the sense that

$$\|u - u_2\|_{L^\infty(ABC)} \leqslant \delta\|u_2'\|_{L^\infty(ABC)} \qquad (6.1)$$

for some (possibly different) second eigenfunction u_2'. What does this say about where the extrema of u are located?

Note that the question behaves well under rescaling of the triangle ABC, so in order to maximise the symmetry we will take ABC to lie in the plane

$$\Pi := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$$

with vertices $A := (0, 0, 0)$, $B := (1, -1, 0)$, $C := (1, 0, -1)$. This is an equilateral triangle of sidelength $\sqrt{2}$ and area $\sqrt{3}/2$. To compute the Laplacian of a function

[1] One may be able to recover this loss of $\sqrt{2}$ with a more complicated analysis if necessary.

$f : ABC \to \mathbb{R}$, one can extend this function first to Π in some arbitrary fashion, and then to $\mathbb{R}^3$ by declaring the function to be constant in the normal direction $(1, 1, 1)$, in which case the three-dimensional Euclidean Laplacian $\Delta_{\mathbb{R}^3}$ coincides on the interior of ABC with the Laplacian on ABC (as can be seen by working in a suitable orthonormal basis of $\mathbb{R}^3$ that includes a unit normal to Π).

Let us first work out what the Neumann eigenfunctions of ABC are. By reflection, a function on ABC with Neumann data can be extended to a function on Π that is periodic with periods $(2, -1, -1)$, $(-1, 2, -1)$, $(-1, -1, 2)$, and is invariant with respect to rotations by 120 degrees around the origin (i.e. $(x, y, z) \mapsto (y, z, x)$ and $(x, y, z) \mapsto (z, x, y)$), and also the reflections $(x, y, z) \mapsto (-y, -x, -z)$, $(x, y, z) \mapsto (-x, -z, -y)$, $(x, y, z) \mapsto (-z, -y, -x)$. After extending invariantly along $(1, 1, 1)$, a function on Π with periods $(2, -1, -1)$, $(-1, 2, -1)$, $(-1, -1, 2)$ becomes a function on $\mathbb{R}^3$ with periods $(3, 0, 0), (0, 3, 0), (0, 0, 3)$ and invariant along $(1, 1, 1)$. From Fourier analysis, the functions on Π with the periods $(2, -1, -1)$, $(-1, 2, -1)$, $(-1, -1, 2)$ can thus be decomposed into plane waves $(x, y, z) \mapsto e^{2\pi i(ax+by+cz)/3}$ with a, b, c integers with $a + b + c = 0$. Putting back in the rotation and reflection symmetry, we see that an orthogonal basis of $L^2(ABC)$ is then given by the complex functions

$$e^{2\pi i(ax+by+cz)/3}+e^{2\pi i(bx+ay+cz)/3}+e^{2\pi i(cx+ay+bz)/3}+e^{2\pi i(-bx-ay-cz)/3}+e^{2\pi i(-ax-cy-bz)/3}+e^{2\pi i(-cx-by-az)/3}$$

for integers a, b, c summing to zero. This is an eigenfunctoin of the Laplacian Δ with eigenvalue $\frac{4\pi^2}{9}(a^2 + b^2 + c^2)$. Thus one has a repeated second eigenvalue $\lambda_2 = \lambda_3 = \frac{8\pi^2}{9}$ spanned by the complex eigenfunction

$$e^{2\pi i(x-y)/3} + e^{2\pi i(y-z)/3} + e^{2\pi i(z-x)/3}$$

and its complex conjugate Note that this function has a mean square of $\sqrt{3}$ and so has an $L^2(ABC)$ norm of $\sqrt{3}(\sqrt{3}/2)^{1/2}$; to normalise in L^2 norm, we would thus have

$$2^{1/4}3^{-3/4}(e^{2\pi i(x-y)/3} + e^{2\pi i(y-z)/3} + e^{2\pi i(z-x)/3})$$

and the real L^2-normalised eigenfunctions take the form

$$u_\theta := 2^{3/4}3^{-3/4}\Re e^{i\theta}(e^{2\pi i(x-y)/3} + e^{2\pi i(y-z)/3} + e^{2\pi i(z-x)/3})$$

for an arbitrary phase θ. In particular we see that

$$\|u_\theta\|_{L^\infty(ABC)} \leqslant 2^{3/4}3^{-3/4} \times 3$$

and so (6.1) can be written as the statement that

$$\|u - u_\theta\|_{L^\infty(ABC)} \leqslant 2^{3/4}3^{1/4}\delta$$

for some θ.

Suppose that u attains an extremum at some point P in ABC but not at the vertices, then

$$|u(P)| > |u(A)|$$

and thus by the triangle inequality

$$|u_\theta(P)| > |u_\theta(A)| - 2^{7/4}3^{1/4}\delta.$$

Writing $P = (x, y, z)$, we conclude that

$$|\Re e^{i\theta}(e^{2\pi i(x-y)/3} + e^{2\pi i(y-z)/3} + e^{2\pi i(z-x)/3})| > |\Re 3e^{i\theta}| - 6\delta.$$

Replacing A by B or C, we similarly obtain

$$|\Re e^{i\theta}(e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3})| > |\Re 3e^{4\pi i/3}e^{i\theta}| - 6\delta$$

and

$$|\Re e^{i\theta}(e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3})| > |\Re 3e^{2\pi i/3}e^{i\theta}| - 6\delta.$$

Suppose that $\Re e^{i\theta}(e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3})$ is non-negative. We then have

$$\Re(e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3}+6\delta e^{-i\theta})e^{i\theta} > \max(\Re 3e^{i\theta}, 3e^{4\pi i/3}e^{i\theta}, 3e^{2\pi i/3}e^{i\theta})$$

and thus $e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3}+6\delta e^{-i\theta}$ lies outside the triangle with vertices 3, $3e^{4\pi i/3}$, $3e^{2\pi i/3}$, and so (by elementary trigonometry) $e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3}$ cannot lie in the triangle with vertices $(3-12\delta)$, $(3-12\delta)e^{4\pi i/3}$, $(3-12\delta)e^{2\pi i/3}$. Similarly when $\Re e^{i\theta}(e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3})$ is non-positive. To put it another way, we must have

$$\Re(e^{2\pi ik/3}(e^{2\pi i(x-y)/3}+e^{2\pi i(y-z)/3}+e^{2\pi i(z-x)/3})) \leqslant -\frac{3}{2}+6\delta \tag{6.2}$$

for some $k=0,1,2$. For δ small, this forces (x,y,z) to be close to one of the three corners A,B,C. For instance, in the $k=0$ case, if we rewrite $(x,y,z)=(x,\frac{-x-t}{2},\frac{-x+t}{2})$ for some $0\leqslant x\leqslant 1$ and $-x\leqslant t\leqslant x$, we see that

$$\Re(e^{2\pi i(x/2+t/6)}+e^{-2\pi it/3}+e^{2\pi i(-x/2+t/6)}) \leqslant -\frac{3}{2}+6\delta \tag{6.3}$$

or equivalently

$$2\cos(\pi t/3)\cos(\pi x)+\cos(2\pi t/3) \leqslant -\frac{3}{2}+6\delta.$$

Observe that $2\cos(\pi t/3)$ is positive, so

$$2\cos(\pi t/3)\cos(\pi x)+\cos(2\pi t/3) \geqslant \cos(2\pi t/3)-2\cos(\pi t/3).$$

Elementary calculus shows that $\cos(2\pi t/3)-2\cos(\pi t/3)$ decreases from -1 to $-3/2$ as t goes from 0 to 1, and is even. Thus for δ small, we see that (6.3) can only occur when $|t|$ is close to 1, which forces x close to 1 also, so that (x,y,z) is close to B or C. Similarly for other values of k. With numerical evaluation of the function in the left-hand side of (6.2) one can presumably get quite a precise bound on how close (x,y,z) is to A,B,C in terms of δ.