1. A Sobolev inequality

**Lemma 1.1.** Let $ABCD$ be a parallelogram, and let $u$ be a $C^2$ function on $ABCD$. Let $\alpha$ be the angle subtended by $A$. Then

$$|u(A) - u(B) + u(C) - u(D)| \leq \frac{1}{\sin(\alpha)} \int_{ABCD} |\nabla^2 u|.$$

**Proof.** We may normalise $A = 0$, so that $C = B + D$. From two applications of the fundamental theorem of calculus one has

$$\int_0^1 \int_0^1 \partial_{st} u(sB + tD) = u(A) - u(B) + u(C) - u(D).$$

The left-hand side can be rewritten as

$$\frac{1}{|ABCD|} \int_{ABCD} (B \cdot \nabla)(D \cdot \nabla)u.$$

Since

$$|ABCD| = |B||D| \sin(\alpha)$$

the claim follows.

**Lemma 1.2.** Let $ABC$ be a triangle with angles $\alpha, \beta, \gamma$, and let $u$ be a $C^2$ function on $ABC$ that obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of $ABC$. Then for any $P \in ABC$, one has

$$|u(P) - u(Q)| \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.$$

**Proof.** Let $P \in ABC$. Let $D, E \in AB, F, G \in BC, H, I \in AC$ be the points such that $ADXI, BFXE, CHXG$ are parallelograms; thus $D, G$ are the intersections with $AB, BC$ respectively of the line through $X$ parallel to $AC$, and so forth. Then from the preceding lemma one has

$$|u(X) + u(A) - u(D) - u(I)| \leq \frac{1}{\sin(\alpha)} \int_{ADXI} |\nabla^2 u|$$

$$|u(X) + u(B) - u(F) - u(E)| \leq \frac{1}{\sin(\beta)} \int_{BFXE} |\nabla^2 u|$$

$$|u(X) + u(C) - u(H) - u(G)| \leq \frac{1}{\sin(\gamma)} \int_{CHXG} |\nabla^2 u|.$$
Also, by reflecting the triangles $DEX, FGX, XHI$ across the Neumann boundary and using the previous lemma, we see that

$$\|u(X) + u(Y) - u(D) - u(E)\| \leq \frac{2}{\sin(\gamma)} \int_{DE} |\nabla^2 u|$$

$$\|u(X) + u(Y) - u(F) - u(G)\| \leq \frac{2}{\sin(\alpha)} \int_{FG} |\nabla^2 u|$$

$$\|u(X) + u(Y) - u(H) - u(I)\| \leq \frac{2}{\sin(\beta)} \int_{XH} |\nabla^2 u|.$$ 

Summing the latter three combinations of $u$ and subtracting the former three using the triangle inequality, we conclude that

$$|3u(X) - u(A) - u(B) - u(C)| \leq \frac{2}{\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.$$ 

A similar inequality holds with $X$ replaced by $Y$. Subtracting the two inequalities, we obtain the claim.

**Corollary 1.3.** Let $ABC$ be a triangle with angles $\alpha, \beta, \gamma$, and let $u$ be a $C^2$ function on $ABC$ which is smooth up to the boundary except possibly at the vertices $A, B, C$, and which obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of $ABC$, and has mean zero on $ABC$. Then

$$\|u\|_{L^\infty(ABC)} \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \|\Delta u\|_{L^2(ABC)}.$$

**Proof.** If $u$ has mean zero, then $\|u\|_{L^\infty(ABC)}$ is bounded by $|u(P) - u(Q)|$ for some $P, Q \in ABC$. From the previous lemma we thus have

$$\|u\|_{L^\infty(ABC)} \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \|\nabla^2 u\|_{L^2(ABC)}.$$

It will thus suffice to show the Bochner-Weitzenbock identity

$$\int_{ABC} |\nabla^2 u|^2 = \int_{ABC} |\Delta u|^2.$$ 

But this can be accomplished by two integration by parts, using the smoothness and Neumann boundary hypotheses on $u$ (and a regularisation argument if necessary to cut away from the vertices) more details needed here.

**2. Schwarz-Christoffel**

Let $0 < \alpha, \beta, \gamma < \pi$ be angles adding up to $\pi$, then we can define a Schwarz-Christoffel map $\Phi_{\alpha, \beta} : \mathbb{H} \to ABC$ from the half-plane $\mathbb{H} := \{z : \Im(z) > 0\}$ to a triangle $ABC$ with angles $\alpha, \beta, \gamma$ by the formula

$$\Phi_{\alpha, \beta}(z) := \int_0^z \frac{d\zeta}{\zeta^{1-\alpha/\pi}(1 - \zeta)^{1-\beta/\pi}}.$$
where the integral is over any contour from 0 to \( z \) in \( \mathbb{H} \), and one chooses the branch cut to make both factors in the denominator positive real on the interval \([0, 1]\). Thus the vertices of the triangle are given by

\[
A := \Phi_{\alpha, \beta}(0) = 0 \\
B := \Phi_{\alpha, \beta}(1) = \int_0^1 \frac{dt}{t^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} = \frac{\Gamma((\alpha + \beta)/\pi)}{\Gamma((\alpha + \beta)/\pi)} \\
C := \Phi_{\alpha, \beta}(\infty) = -e^{i\alpha} \int_{-\infty}^0 \frac{dt}{|t|^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} = e^{i\alpha} \frac{\Gamma((\alpha + \beta)/\pi)}{\Gamma((\alpha + \beta)/\pi)}
\]

where we have used the beta function identity

\[
\int_0^1 \frac{dt}{t^{1-x}(1-t)^{1-y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]

and the changes of variable \( s = 1 - t, \; v = 1/s \). In particular, the area of the triangle \( ABC \) can be expressed as

\[
|ABC| = \frac{1}{2} |B||C| \sin(\alpha) = \frac{\Gamma((\alpha + \beta)/\pi)}{2 \Gamma((\alpha + \beta)/\pi)} \sin(\alpha)
\]

which can be simplified using the formula \( \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \) as the more symmetric expression

\[
|ABC| = \frac{1}{2\pi^2} \Gamma((\alpha/\pi)^2 \Gamma(\beta/\pi)^2 \Gamma(\gamma/\pi)^2 \sin(\alpha) \sin(\beta) \sin(\gamma)). \tag{2.1}
\]

We write

\[
|\Phi'_{\alpha, \beta}(z)| = e^{\omega(z)}
\]

where \( \omega = \omega_{\alpha, \beta} \) is the harmonic function

\[
\omega(z) := \left( \frac{\alpha}{\pi} - 1 \right) \log |z| + \left( \frac{\beta}{\pi} - 1 \right) \log |1-z|. \tag{2.2}
\]

If \( u : ABC \to \mathbb{R} \) is a smooth function, and \( \tilde{u} : \mathbb{H} \to \mathbb{R} \) is its pullback to the half-plane \( \mathbb{H} \) defined by

\[
\tilde{u} := u \circ \Phi_{\alpha, \beta}
\]

then we have

\[
\int_{ABC} u = \int_{\mathbb{H}} e^{2\omega} \tilde{u}.
\]
In a similar vein we have the conformal invariance of the two-dimensional Dirichlet energy

\[ \int_{ABC} |\nabla u|^2 = \int_{H} |\nabla \tilde{u}|^2 \]

and the conformal transformation of the Laplacian:

\[ \Delta \tilde{u}(z) = e^{2\omega} \Delta u. \]

In particular, the Rayleigh quotient

\[ \int_{ABC} |\nabla u|^2 / \int_{ABC} |u|^2 \]

with mean zero condition \( \int_{ABC} u = 0 \) becomes, when pulled back to \( H \), the Rayleigh quotient

\[ \int_{H} |\nabla \tilde{u}|^2 / \int_{H} e^{2\omega}|\tilde{u}|^2 \]

with mean zero condition \( \int_{H} e^{2\omega} \tilde{u} = 0 \).

Let \( u_2, u_3, \ldots \) be an \( L^2 \)-normalised eigenbasis for the Neumann Laplacian \( -\Delta \) on \( ABC \) with eigenvalues \( \lambda_2 \leq \lambda_3 \leq \ldots \), thus

\[ -\Delta u_k = \lambda_k u_k \]
on \( ABC \) with Neumann boundary data

\[ n \cdot \nabla u_k = 0 \]

and orthonormality

\[ \int_{ABC} u_j u_k = \delta_{jk} \]

and mean zero condition

\[ \int_{ABC} u_j = 0. \]

One can show that when \( ABC \) is acute-angled, these eigenfunctions are smooth except possibly at the vertices \( A, B, C \), and are uniformly \( C^2 \). add details here

Pulling all this back to \( H \), we obtain transformed eigenfunctions \( \tilde{u}_2, \tilde{u}_3, \ldots \) on \( H \) to the conformal eigenfunction equation

\[ -\Delta \tilde{u}_k = \lambda_k e^{2\omega} \tilde{u}_k \quad (2.3) \]
on \( H \) with Neumann boundary data

\[ n \cdot \nabla \tilde{u}_k = 0 \quad (2.4) \]

and orthonormality

\[ \int_{H} e^{2\omega} \tilde{u}_j \tilde{u}_k = \delta_{jk} \quad (2.5) \]

and mean zero condition

\[ \int_{H} e^{2\omega} \tilde{u}_j = 0. \quad (2.6) \]
Now suppose that we vary the angle parameters $\alpha, \beta, \gamma$ smoothly with respect to some time parameter $t$, thus also varying the triangles $ABC$, eigenfunctions $u_k$ and transformed eigenfunctions $\tilde{u}_k$, eigenvalues $\lambda_k$, and conformal factor $\omega$. We will use dots to indicate time differentiation, thus for instance $\dot{\alpha} = \frac{d}{dt}\alpha$. Let us formally suppose that all of the above data vary smoothly (or at least $C^1$) in time we will eventually need to justify this, of course. Since $\alpha + \beta + \gamma = \pi$, we have

$$\dot{\alpha} + \dot{\beta} + \dot{\gamma} = 0.$$  

The variation $\dot{\omega}$ of the conformal factor is explicitly computable from (2.2) as being a logarithmic weight:

$$\dot{\omega} = \frac{\dot{\alpha}}{\pi} \log |z| + \frac{\dot{\beta}}{\pi} \log |1 - z|.$$ 

Next, by (formally) differentiating (2.3) we obtain an equation for the variation $\dot{u}_k$ of the $k^{th}$ eigenfunction:

$$-\Delta \dot{u}_k = \lambda_k e^{2\omega} \dot{u}_k + 2\lambda_k \dot{\omega} e^{2\omega} \tilde{u}_k + \lambda_k e^{2\omega} \ddot{u}_k.$$  \hspace{1cm} (2.7)

To solve this equation for $\dot{u}_k$, we observe from differentiating (2.4), (2.5), (2.6) that

$$\int_H e^{2\omega} \dot{u}_k = 0$$  \hspace{1cm} (2.8)

and

$$\int_H e^{2\omega} \ddot{u}_k \dot{u}_k = 0$$

and

$$n \cdot \nabla \dot{u}_k = 0.$$ 

By eigenfunction expansion, we thus have

$$\dot{u}_k = \sum_{l \neq k} (\int_H e^{2\omega} \tilde{u}_l \dot{u}_k) \tilde{u}_l$$  \hspace{1cm} (2.9)

in a suitable sense ($L^2$ with weight $e^{2\omega}$). Now we evaluate the expression in parentheses. Integrating (2.7) against $\tilde{u}_l$ and using (2.5) reveals that

$$-\int_H \Delta \dot{u}_k \tilde{u}_l = 2\lambda_k \int_H \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l + \lambda_k \int_H e^{2\omega} \ddot{u}_k \tilde{u}_l.$$  \hspace{1cm} (2.10)

By Green’s theorem and the Neumann conditions on $\dot{u}_k$ and $\tilde{u}_l$, the left-hand side is

$$-\int_H \dot{u}_k \Delta \tilde{u}_l$$

which by (2.3) is equal to

$$\lambda_l \int_H e^{2\omega} \dot{u}_k \tilde{u}_l.$$ 

Inserting this into (2.10) we see that

$$\int_H e^{2\omega} \ddot{u}_k \tilde{u}_l = \frac{2\lambda_k}{\lambda_l - \lambda_k} \int_H \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l$$
and thus by (2.9)
\[ \dot{u}_k = \sum_{l \neq k} \left( \frac{2\lambda_l}{\lambda_l - \lambda_k} \int_{\mathcal{H}} \omega e^{2\omega} \dot{u}_k \dot{u}_l \right) \dot{u}_l. \]  
(2.11)

We can take Laplacians and conclude that
\[ -\Delta \dot{u}_k = e^{2\omega} \sum_{l \neq k} \left( \frac{2\lambda_l \lambda_l}{\lambda_l - \lambda_k} \int_{\mathcal{H}} \omega e^{2\omega} \dot{u}_k \dot{u}_l \right) \dot{u}_l. \]

Set \( k = 2 \), then \( \frac{2\lambda_k \lambda_k}{\lambda_k - \lambda_2} \) is bounded in magnitude by \( \frac{2\lambda_2 \lambda_2}{\lambda_2 - \lambda_2} \). From the orthonormality (2.5) and the Bessel inequality, we conclude that
\[ \left( \int_{\mathcal{H}} e^{-2\omega} |\Delta \dot{u}_2|^2 \right)^{1/2} \leq \frac{2\lambda_2 \lambda_2}{\lambda_2 - \lambda_2} \left( \int_{\mathcal{H}} |\omega|^2 e^{2\omega} \dot{u}_2^2 \right)^{1/2}. \]

If we change coordinates by writing \( \dot{u} = \dot{u} \circ \Phi \)
we conclude that
\[ \left( \int_{ABC} |\Delta \dot{u}_2|^2 \right)^{1/2} \leq \frac{2\lambda_2 \lambda_2}{\lambda_2 - \lambda_2} \left( \int_{\mathcal{H}} |\omega|^2 e^{2\omega} \dot{u}_2^2 \right)^{1/2}. \]

Also, \( \dot{u}_2 \) has mean zero on \( ABC \) by (2.8). We conclude from Corollary 1.3 that
\[ \| \dot{u}_2 \|_{L^\infty} \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma)) |ABC|^{1/2}} \frac{2\lambda_2 \lambda_2}{\lambda_2 - \lambda_2} \left( \int_{\mathcal{H}} |\omega|^2 e^{2\omega} \dot{u}_2^2 \right)^{1/2}. \]

Pulling back to \( \mathcal{H} \), and estimating \( \dot{u}_2 \) in \( L^\infty \) norm, we conclude that
\[ \| \dot{u}_2 \|_{L^\infty(\mathcal{H})} \leq X \| \dot{u}_2 \|_{L^\infty(\mathcal{H})} \]
where \( X \) is the explicit (but somewhat messy) quantity
\[ X := \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma)) |ABC|^{1/2}} \frac{2\lambda_2 \lambda_2}{\lambda_2 - \lambda_2} \left( \int_{\mathcal{H}} |\omega|^2 e^{2\omega} \right)^{1/2}. \]

This gives stability of the second eigenfunction in \( L^\infty \) norm, as long as there is an eigenvalue gap \( \lambda_2 - \lambda_2 > 0 \).

One can compute one factor in the quantity \( X \) as follows. Observe that
\[ \int_{\mathcal{H}} e^{2\omega} = |ABC| = \frac{1}{2\pi^2} \Gamma(\alpha/\pi)^2 \Gamma(\beta/\pi)^2 \Gamma(\gamma/\pi)^2 \sin(\alpha) \sin(\beta) \sin(\gamma). \]

If we view \( \alpha, \beta, \gamma \) (and hence \( \omega \)) as varying linearly in time, and differentiate the above equation under the integral sign twice in time, we conclude that
\[ 4 \int_{\mathcal{H}} |\omega|^2 e^{2\omega} = \frac{d^2}{dt^2} \int_{\mathcal{H}} e^{2\omega} \]
\[ = (\dot{\omega}^2 \frac{\partial^2}{\partial \alpha^2} + 2\dot{\alpha} \dot{\beta} \frac{\partial^2}{\partial \alpha \partial \beta} + 2\dot{\alpha} \dot{\gamma} \frac{\partial^2}{\partial \alpha \partial \gamma} \]
\[ + \dot{\beta}^2 \frac{\partial^2}{\partial \beta^2} + 2\dot{\beta} \dot{\gamma} \frac{\partial^2}{\partial \beta \partial \gamma} + \dot{\gamma}^2 \frac{\partial^2}{\partial \gamma^2} \]
\[ + \left( \frac{1}{2\pi^2} \Gamma(\alpha/\pi)^2 \Gamma(\beta/\pi)^2 \Gamma(\gamma/\pi)^2 \sin(\alpha) \sin(\beta) \sin(\gamma) \right). \]
This, in principle, expresses the factor \( \left( \int_H |\dot{\omega}|^2 e^{2\omega} \right)^{1/2} \) in \( X \) as an explicit combination of trigonometric functions, gamma functions, and the first two derivatives of the gamma function. However, this formula is somewhat messy, to say the least.

3. AN EXPLICIT EXAMPLE

Suppose we take the isosceles right-angled triangle

\[ \alpha = \pi/2; \beta = \gamma = \pi/4 \]

and move along the space of right-angled triangles by taking

\[ \dot{\alpha} = 0; \dot{\beta} = 1; \dot{\gamma} = -1. \]

In this example we have

\[ \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} = \frac{4\sqrt{2}}{3} \approx 1.8856 \]

and

\[ |ABC|^{1/2} = \frac{1}{2\pi} \Gamma(1/2)\Gamma(1/4)^2 \approx 3.70815 \]

To put it another way, the sidelength \( AB = AC \) is given by

\[ |AB| = |AC| = \frac{1}{\sqrt{2\pi}} \Gamma(1/2)\Gamma(1/4)^2 \approx 5.24412. \]

We have

\[ \lambda_2 = \frac{\pi^2}{|AB|^2} \]

and

\[ \lambda_3 = \frac{2\pi^2}{|AB|^2} \]

and so

\[ \frac{2\lambda_2 \lambda_3}{\lambda_3 - \lambda_2} = \frac{4\pi^2}{|AB|^2} = 7.52814. \]

MAPLE tells me that

\[ \frac{d^2}{dt^2} |ABC| \approx 40.836 \]

and so

\[ \int_H |\dot{\omega}|^2 e^{2\omega} \approx 10.209 \]

and so

\[ X \approx 22.34. \]
In the above analysis, we assumed without proof that the eigenfunction \( u \) and eigenvalue \( \lambda \) behaved in a \( C^1 \) fashion with respect to smooth deformation of the triangle. We now justify this assertion.

It is convenient to first work with an affine model rather than a conformal one, using a fixed reference triangle \( \Omega_0 \) instead of the half-plane \( \mathbb{H} \) as the reference domain, and affine maps instead of Schwarz-Christoffel maps as the transformation maps. This is in order to keep the transformation maps smooth at the vertices; it comes at the cost of making the Neumann condition inhomogeneous. After we establish smoothness for this model, we will then change coordinates to the conformal model.

Namely, suppose one has a smooth family of triangles \( ABC = ABC(t) \) with vertices \( A(t), B(t), C(t) \) depending smoothly on a time parameter \( t \) (as is the case in the preceding discussion). We isolate the time zero triangle \( \Omega_0 := ABC(0) \) as the reference domain, and view all the other triangles as affine images \( \Omega(t) = F(t)(\Omega_0) \) of the reference triangle for some affine transformations \( F(t) : \mathbb{R}^2 \to \mathbb{R}^2 \) depending smoothly on \( t \). Actually, we may normalise \( A(t) = 0 \) (say), so that the \( F(t) \) are linear instead of affine.

Pulling the Rayleigh quotient from \( \Omega(t) \) back to \( \Omega_0 \), we see that the second eigenvalue \( \lambda_2(t) \) comes from minimising the functional

\[
\int_{\Omega_0} |F^{-1}(t) \nabla u|^2 \int_{\Omega_0} |u|^2
\]

among functions \( u \in L^2(\Omega_0) \) of mean zero. Let us write \( u_2(t) \) for a minimiser of this functional with unit norm, thus

\[
\int_{\Omega_0} |u_2(t)|^2 = 1
\]

and

\[
\int_{\Omega_0} |F^{-1}(t) \nabla u_2(t)|^2 = \lambda_2(t);
\]

as long as the second eigenvalue is simple, this uniquely determines \( u_2(t) \) up to sign. The function \( u_2 \) is of course the second Neumann eigenfunction of \( \Omega(t) \), pulled back to \( \Omega_0 \).

We now compare \( u_2(t) \) with \( u_2(0) \) for \( t \) small, assuming an eigenvalue gap \( \lambda_2(0) < \lambda_3(0) \). Let us write

\[
u_2(t) = \cos \theta(t) u_2(0) + \sin \theta(t) v(t)\]

for some angle \( \theta(t) \) and some \( v(t) \) orthogonal to both \( u_2(0) \) and 1, and of unit \( L^2 \) norm. By reflection we may also assume that \( |\theta(t)| \leq \pi/2 \). Because \( u_2(t) \) achieves the minimum of the Rayleigh quotient, we see that

\[
\int_{\Omega_0} |F^{-1} \nabla u_2(t)|^2 \leq \int_{\Omega_0} |F^{-1} \nabla u_2(0)|^2.
\]
The left-hand side can be expanded as
\[
\cos^2 \theta(t) \int_{\Omega_0} |F^{-1}(t)\nabla u_2(0)|^2 + 2 \cos \theta(t) \sin \theta(t) \int_{\Omega_0} (F^{-1}(t))^* F^{-1}(t) \nabla u_2(0) \cdot \nabla v(t) + \sin^2 \theta(t) \int_{\Omega_0} |F^{-1}(t)|^2
\]
and thus, either \( \theta(t) = 0 \) or
\[
2 \cos \theta(t) \int_{\Omega_0} (F^{-1}(t))^* F^{-1}(t) \nabla u_2(0) \cdot \nabla v(t) + \sin \theta(t) \int_{\Omega_0} |F^{-1}(t)\nabla v(0)|^2 \leq \sin \theta(t) \int_{\Omega_0} |F^{-1}(t)\nabla u_2(0)|^2.
\]
Note that as \( F \) depends smoothly on \( t \), one has
\[
\int_{\Omega_0} |F^{-1}(t)\nabla u_2(0)|^2 = \lambda_2(0) + O(|t|)
\]
and similarly
\[
\int_{\Omega_0} |F^{-1}(t)\nabla v(0)|^2 \geq (1 - O(|t|)) \int_{\Omega_0} |\nabla v(0)|^2 \geq (1 - O(|t|))\lambda_3(0)
\]
thanks to the spectral theorem and the normalisation of \( v \); here the implied constants are allowed to depend on everything except \( t \). Thus, for sufficiently small \( t \), we conclude that
\[
|\sin \theta(t)| \int_{\Omega_0} |F^{-1}(t)\nabla v(0)|^2 \ll \int_{\Omega_0} (F^{-1}(t))^* F^{-1}(t) \nabla u_2(0) \cdot \nabla v(t)|.
\]
The eigenfunction \( u_2(0) \) obeys the eigenfunction equation \( -\Delta u_2(0) = \lambda_2(0)u_2(0) \) with Neumann boundary condition \( n \cdot \nabla u_2(0) = 0 \), and is bounded in \( H^2 \). In particular, by integration by parts
\[
\int_{\Omega_0} \nabla u_2(0) \cdot \nabla v(t) = \int_{\Omega_0} \lambda_2(0)u_2(0)v(t) = 0
\]
and so
\[
\int_{\Omega_0} (F^{-1}(t))^* F^{-1}(t) \nabla u_2(0) \cdot \nabla v(t) = O(|t|) \int_{\Omega_0} |\nabla u_2(0)||\nabla v(t)|
\]
which by Cauchy-Schwarz gives
\[
\int_{\Omega_0} (F^{-1}(t))^* F^{-1}(t) \nabla u_2(0) \cdot \nabla v(t) = O(|t|) \int_{\Omega_0} |F^{-1}(t)\nabla v(0)|^2)^{1/2}.
\]
We thus have
\[
\sin \theta(t) \int_{\Omega_0} |F^{-1}(t)\nabla v(0)|^2)^{1/2} = O(|t|)
\]
and thus \( \theta(t) = O(|t|) \) and
\[
\|\sin \theta(t)v\|_{H^1(\Omega_0)} = O(|t|).
\]
In particular, we have the Lipschitz bound
\[
\|u_2(t) - u_2(0)\|_{H^1(\Omega_0)} = O(|t|)
\]
for \( t \) small enough. Comparing Rayleigh quotients then gives
\[
|\lambda_2(t) - \lambda_2(0)| = O(|t|)
\]
again for \( t \) small enough.

Now we move back to the conformal picture. Taking into account the difference between the affine coordinate transformations and the Schwarz-Christoffel transformations, the
$H^1$ bound (4.1) implies that the eigenfunctions $\tilde{u}_2(t)$, $\tilde{u}_2(0)$ obey an $L^2$ estimate of the form

$$\left( \int_{\mathbb{H}} |\tilde{u}_2(t, z) - \tilde{u}_2(0, z)|^2 e^{2\omega(0)} \right)^{1/2} = O(|t|).$$

This can be established by using uniform $C^0$ bounds on $u_2(t)$ near vertices to handle the regions within $O(|t|^K)$ of the vertices $A, B, C$ for some large constant $K$, and using (4.1) and a smooth deformation in $t$ to control the remainder. If we write $\tilde{u}_2(t) = \tilde{u}_2(0) + t\tilde{v}(t)$, we thus have

$$\|\tilde{v}(t)e^{\omega(0)}\|_{L^2(\mathbb{H})} = O(1)$$

with $\tilde{v}(t)$ obeying Neumann boundary conditions. The same argument also gives the variant bounds

$$\|\tilde{v}(t)(e^{\omega(t)} - e^{\omega(0)})\|_{L^2(\mathbb{H})} = o(1).$$

Also, from the eigenfunction equation

$$-\Delta \tilde{u}_2(t) = -\lambda_2(t) e^{2\omega(t)} \tilde{u}_2(t)$$

and writing $\lambda_2(t) = \lambda_2(0) + t\gamma(t)$, $\omega(t) = \omega(0) + t\sigma(t)$, we see that

$$-\Delta \tilde{v}(t) = -e^{2\omega_0}((\lambda_2(0)+t\gamma(t)) e^{2\sigma(t)} - 1)(\tilde{u}_2(0) + t\tilde{v}(t)) + \gamma(t) \tilde{u}_2(0) + \lambda_2(0) \tilde{v}(t) + t\gamma(t) \tilde{v}(t).$$

Since $\gamma(t) = O(1)$, the previous bounds on $\tilde{v}$ give

$$\|e^{-\omega(t)}\|_{L^2(\mathbb{H})} = O(1)$$

and in particular (on integrating this against $\tilde{v}(t)e^{\omega(0)}$)

$$\|\nabla \tilde{v}\|_{L^2(\mathbb{H})} = O(1).$$

From this and Hardy’s inequality one obtains (after some calculation) that

$$(-\Delta - \lambda_2 e^{2\omega(t)})\tilde{v}(t) = -e^{2\omega(t)}(2\lambda_2(0)\sigma(0)\tilde{u}_2(0) + \gamma(t) \tilde{u}_2(0) + e(t))$$

where $\|e(t)\|_{L^2(\mathbb{H})} = o(1)$. Integrating this against $\tilde{u}_2(0)$, we obtain that

$$0 = -2\lambda_2(0)\sigma(0) \int_{\mathbb{H}} e^{2\omega(t)} \tilde{u}_2(0)^2 + \gamma(t) + o(1)$$

which among other things shows that $\gamma(t)$ is continuous at $t = 0$, and we now have

$$(-\Delta - \lambda_2 e^{2\omega(t)})\tilde{v}(t) = -e^{2\omega(t)}(2\lambda_2(0)\sigma(0)\tilde{u}_2(0) + \gamma(t) \tilde{u}_2(0) + e'(t))$$

where $\|e(t)\|_{L^2(\mathbb{H})} = o(1)$. Solving this inhomogeneous eigenfunction equation, we see that the component of $\tilde{v}(t)$ orthogonal to $\tilde{u}_2(0)$ in $L^2(\mathbb{H}, e^{2\omega(t)})$ is continuous at $t = 0$ in the $L^2(\mathbb{H}, e^{2\omega(t)})$ norm. As for the component parallel to $\tilde{u}_2(0)$, we use the normalisation

$$\int_{\mathbb{H}} |\tilde{u}_2(t)|^2 e^{2\omega(t)} = \int_{\mathbb{H}} |\tilde{u}_2(0)|^2 e^{2\omega_0} = 1$$

which we rewrite using $\tilde{u}_2(t) = \tilde{u}_2(0) + t\tilde{v}(t)$ as

$$2 \int_{\mathbb{H}} \tilde{u}_2(0)\tilde{v}(0) e^{2\omega(t)} = \int_{\mathbb{H}} |\tilde{u}_2(0)|^2 e^{2\omega(t)} - \int_{\mathbb{H}} |\tilde{u}_2(0)|^2 e^{2\omega_0} + 2 \int_{\mathbb{H}} \tilde{u}_2(0)\tilde{v}(0)(e^{2\omega(t)} - e^{2\omega_0}) + t \int_{\mathbb{H}} |\tilde{v}(0)|^2 e^{2\omega(t)}. $$

The terms in the right-hand side can be evaluated to be $-2\int_{\mathbb{H}} |\tilde{u}_2(0)|^2\omega_0 + o(1)$ (for the first integral on the right-hand side we have to treat the region very close to the vertices using $C^0$ bounds on $\tilde{u}_2$). Thus we see that this component also depends continuously
on $t$. This gives differentiability of $\tilde{u}_2$ and $\lambda_2$ in $t$ (and even gives the correct explicit formula for the derivative).