STABILITY THEORY FOR NEUMANN EIGENFUNCTIONS

1. A Sobolev inequality

Lemma 1.1. *Let $ABCD$ be a parallelogram, and let u be a C^2 function on $ABCD$. Let α be the angle subtended by A. Then*

$$|u(A) - u(B) + u(C) - u(D)| \leqslant \frac{1}{\sin(\alpha)} \int_{ABCD} |\nabla^2 u|.$$

Proof. We may normalise $A = 0$, so that $C = B + D$. From two applications of the fundamental theorem of calculus one has

$$\int_0^1 \int_0^1 \partial_{st} u(sB + tD) = u(A) - u(B) + u(C) - u(D).$$

The left-hand side can be rewritten as

$$\frac{1}{|ABCD|} \int_{ABCD} (B \cdot \nabla)(D \cdot \nabla) u.$$

Since

$$|ABCD| = |B||D| \sin(\alpha)$$

the claim follows. □

Lemma 1.2. *Let ABC be a triangle with angles α, β, γ, and let u be a C^2 function on ABC that obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of ABC. Then for any $P \in ABC$, one has*

$$|u(P) - u(Q)| \leqslant \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.$$

Proof. Let $P \in ABC$. Let $D, E \in AB$, $F, G \in BC, H, I \in AC$ be the points such that $ADXI, BFXE, CHXG$ are parallelograms (thus D, G are the intersections with AB, BC respectively of the line through X parallel to AC, and so forth. Then from the preceding lemma one has

$$|u(X) + u(A) - u(D) - u(I)| \leqslant \frac{1}{\sin(\alpha)} \int_{ADXI} |\nabla^2 u|$$

$$|u(X) + u(B) - u(F) - u(E)| \leqslant \frac{1}{\sin(\beta)} \int_{BFXE} |\nabla^2 u|$$

$$|u(X) + u(C) - u(H) - u(G)| \leqslant \frac{1}{\sin(\gamma)} \int_{CHXG} |\nabla^2 u|.$$

Also, by reflecting the triangles DEX, FGX, XHI across the Neumann boundary and using the previous lemma, we see that

$$|u(X) + u(X) - u(D) - u(E)| \leqslant \frac{2}{\sin(\gamma)} \int_{DEX} |\nabla^2 u|$$
$$|u(X) + u(X) - u(F) - u(G)| \leqslant \frac{2}{\sin(\alpha)} \int_{FGX} |\nabla^2 u|$$
$$|u(X) + u(X) - u(H) - u(I)| \leqslant \frac{2}{\sin(\beta)} \int_{XHI} |\nabla^2 u|.$$

Summing the latter three combinations of u and subtracting the former three using the triangle inequality, we conclude that

$$|3u(X) - u(A) - u(B) - u(C)| \leqslant \frac{2}{\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.$$

A similar inequality holds with X replaced by Y. Subtracting the two inequailties, we obtain the claim. □

Corollary 1.3. *Let ABC be a triangle with angles α, β, γ, and let u be a C^2 function on ABC which is smooth up to the boundary except possibly at the vertices A, B, C, and which obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of ABC, and has mean zero on ABC. Then*

$$\|u\|_{L^\infty(ABC)} \leqslant \frac{4}{3\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \|\Delta u\|_{L^2(ABC)}.$$

Proof. If u has mean zero, then $\|u\|_{L^\infty(ABC)}$ is bounded by $|u(P) - u(Q)|$ for some $P, Q \in ABC$. From the previous lemma we thus have

$$\|u\|_{L^\infty(ABC)} \leqslant \frac{4}{3\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \|\nabla^2 u\|_{L^2(ABC)}.$$

It will thus suffice to show the Bochner-Weitzenbock identity

$$\int_{ABC} |\nabla^2 u|^2 = \int_{ABC} |\Delta u|^2.$$

But this can be accomplished by two integration by parts, using the smoothness and Neumann boundary hypotheses on u (and a regularisation argument if necessary to cut away from the vertices) **more details needed here**. □

2. Schwarz-Christoffel

Let $0 < \alpha, \beta, \gamma < \pi$ be angles adding up to π, then we can define a Schwarz-Christoffel map $\Phi_{\alpha,\beta} : \mathbb{H} \to ABC$ from the half-plane $\mathbb{H} := \{z : \Im(z) > 0\}$ to a triangle ABC with angles α, β, γ by the formula

$$\Phi_{\alpha,\beta}(z) := \int_0^z \frac{d\zeta}{\zeta^{1-\alpha/\pi}(1-\zeta)^{1-\beta/\pi}},$$

where the integral is over any contour from 0 to z in $\mathbb{H}$, and one chooses the branch cut to make both factors in the denominator positive real on the interval $[0,1]$. Thus the vertices of the triangle are given by

$$\begin{aligned}
A &:= \Phi_{\alpha,\beta}(0) = 0 \\
B &:= \Phi_{\alpha,\beta}(1) = \int_0^1 \frac{dt}{t^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} = \frac{\Gamma(\alpha/\pi)\Gamma(\beta/\pi)}{\Gamma((\alpha+\beta)/\pi)} \\
C &:= \Phi_{\alpha,\beta}(\infty) = -e^{i\alpha}\int_0^{-\infty} \frac{dt}{|t|^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} \\
&= e^{i\alpha}\int_1^{\infty} \frac{ds}{(s-1)^{1-\alpha/\pi}s^{1-\beta/\pi}} \\
&= e^{i\alpha}\int_0^{1} \frac{dv}{(v-1)^{1-\alpha/\pi}v^{1-\gamma/\pi}} \\
&= e^{i\alpha}\frac{\Gamma(\alpha/\pi)\Gamma(\gamma/\pi)}{\Gamma((\alpha+\gamma)/\pi)}
\end{aligned}$$

where we have used the beta function identity

$$\int_0^1 \frac{dt}{t^{1-x}(1-t)^{1-y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and the changes of variable $s = 1-t$, $v = 1/s$. In particular, the area of the triangle ABC can be expressed as

$$|ABC| = \frac{1}{2}|B||C|\sin(\alpha) = \frac{\Gamma(\alpha/\pi)^2\Gamma(\beta/\pi)\Gamma(\gamma/\pi)}{2\Gamma((\alpha+\beta)/\pi)\Gamma((\alpha+\gamma)/\pi)}\sin(\alpha)$$

which can be simplified using the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ as the more symmetric expression

$$|ABC| = \frac{1}{2\pi^2}\Gamma(\alpha/\pi)^2\Gamma(\beta/\pi)^2\Gamma(\gamma/\pi)^2\sin(\alpha)\sin(\beta)\sin(\gamma). \tag{2.1}$$

We write

$$|\Phi'_{\alpha,\beta}(z)| = e^{\omega(z)}$$

where $\omega = \omega_{\alpha,\beta}$ is the harmonic function

$$\omega(z) := (\frac{\alpha}{\pi} - 1)\log|z| + (\frac{\beta}{\pi} - 1)\log|1-z|. \tag{2.2}$$

If $u : ABC \to \mathbb{R}$ is a smooth function, and $\tilde{u} : \mathbb{H} \to \mathbb{R}$ is its pullback to the half-plane $\mathbb{H}$ defined by

$$\tilde{u} := u \circ \Phi_{\alpha,\beta}$$

then we have

$$\int_{ABC} u = \int_{\mathbb{H}} e^{2\omega}\tilde{u}.$$

In a similar vein we have the conformal invariance of the two-dimensional Dirichlet energy

$$\int_{ABC} |\nabla u|^2 = \int_{\mathbb{H}} |\nabla \tilde{u}|^2$$

and the conformal transformation of the Laplacian:

$$\Delta \tilde{u}(z) = e^{2\omega} \widetilde{\Delta u}.$$

In particular, the Rayleigh quotient

$$\int_{ABC} |\nabla u|^2 / \int_{ABC} |u|^2$$

with mean zero condition $\int_{ABC} u = 0$ becomes, when pulled back to $\mathbb{H}$, the Rayleigh quotient

$$\int_{\mathbb{H}} |\nabla \tilde{u}|^2 / \int_{\mathbb{H}} e^{2\omega} |\tilde{u}|^2$$

with mean zero condition $\int_{\mathbb{H}} e^{2\omega} \tilde{u} = 0$.

Let $u_2, u_3, \dots$ be an L^2-normalised eigenbasis for the Neumann Laplacian $-\Delta$ on ABC with eigenvalues $\lambda_2 \leqslant \lambda_3 \leqslant \dots$, thus

$$-\Delta u_k = \lambda_k u_k$$

on ABC with Neumann boundary data

$$n \cdot \nabla u_k = 0$$

and orthonormality

$$\int_{ABC} u_j u_k = \delta_{jk}$$

and mean zero condition

$$\int_{ABC} u_j = 0.$$

One can show that when ABC is acute-angled, these eigenfunctions are smooth except possibly at the vertices A, B, C, and are uniformly C^2. **add details here**

Pulling all this back to $\mathbb{H}$, we obtain transformed eigenfunctions $\tilde{u}_2, \tilde{u}_3, \dots$ on $\mathbb{H}$ to the conformal eigenfunction equation

$$-\Delta \tilde{u}_k = \lambda_k e^{2\omega} \tilde{u}_k \tag{2.3}$$

on $\mathbb{H}$ with Neumann boundary data

$$n \cdot \nabla \tilde{u}_k = 0 \tag{2.4}$$

and orthonormality

$$\int_{\mathbb{H}} e^{2\omega} \tilde{u}_j \tilde{u}_k = \delta_{jk} \tag{2.5}$$

and mean zero condition

$$\int_{\mathbb{H}} e^{2\omega} \tilde{u}_j = 0. \tag{2.6}$$

Now suppose that we vary the angle parameters α, β, γ smoothly with respect to some time parameter t, thus also varying the triangles ABC, eigenfunctions u_k and transformed eigenfunctions $\tilde{u}_k$, eigenvalues λ_k, and conformal factor ω. We will use dots to indicate time differentiation, thus for instance $\dot{\alpha} = \frac{d}{dt}\alpha$. Let us formally suppose that all of the above data vary smoothly (or at least C^1) in time **we will eventually need to justify this, of course**. Since $\alpha + \beta + \gamma = \pi$, we have

$$\dot{\alpha} + \dot{\beta} + \dot{\gamma} = 0.$$

The variation $\dot{\omega}$ of the conformal factor is explicitly computable from (2.2) as being a logarithmic weight:

$$\dot{\omega} = \frac{\dot{\alpha}}{\pi} \log |z| + \frac{\dot{\beta}}{\pi} \log |1 - z|.$$

Next, by (formally) differentiating (2.3) we obtain an equation for the variation $\dot{\tilde{u}}_k$ of the k^{th} eigenfunction:

$$-\Delta \dot{\tilde{u}}_k = \dot{\lambda}_k e^{2\omega} \tilde{u}_k + 2\lambda_k \dot{\omega} e^{2\omega} \tilde{u}_k + \lambda_k e^{2\omega} \dot{\tilde{u}}_k. \tag{2.7}$$

To solve this equation for $\dot{\tilde{u}}_k$, we observe from differentiating (2.4), (2.5), (2.6) that

$$\int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k = 0 \tag{2.8}$$

and

$$\int_{\mathbb{H}} e^{2\omega} \tilde{u}_k \dot{\tilde{u}}_k = 0$$

and

$$n \cdot \nabla \dot{\tilde{u}}_k = 0.$$

By eigenfunction expansion, we thus have

$$\dot{\tilde{u}}_k = \sum_{l \neq k} (\int_{\mathbb{H}} e^{2\omega} \tilde{u}_l \dot{\tilde{u}}_k) \tilde{u}_l \tag{2.9}$$

in a suitable sense (L^2 with weight e^{ω}). Now we evaluate the expression in parentheses. Integrating (2.7) against $\tilde{u}_l$ and using (2.5) reveals that

$$-\int_{\mathbb{H}} \Delta \dot{\tilde{u}}_k \tilde{u}_l = 2\lambda_k \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l + \lambda_k \int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l. \tag{2.10}$$

By Green's theorem and the Neumann conditions on $\dot{\tilde{u}}_k$ and $\tilde{u}_l$, the left-hand side is

$$-\int_{\mathbb{H}} \dot{\tilde{u}}_k \Delta \tilde{u}_l$$

which by (2.3) is equal to

$$\lambda_l \int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l.$$

Inserting this into (2.10) we see that

$$\int_{\mathbb{H}} e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l = \frac{2\lambda_k}{\lambda_l - \lambda_k} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l$$

and thus by (2.9)

$$\dot{\tilde{u}}_k = \sum_{l \neq k} (\frac{2\lambda_k}{\lambda_l - \lambda_k} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l) \tilde{u}_l. \tag{2.11}$$

We can take Laplacians and conclude that

$$-\Delta \dot{\tilde{u}}_k = e^{2\omega} \sum_{l \neq k} (\frac{2\lambda_k \lambda_l}{\lambda_l - \lambda_k} \int_{\mathbb{H}} \dot{\omega} e^{2\omega} \tilde{u}_k \tilde{u}_l) \tilde{u}_l.$$

Set $k = 2$, then $\frac{2\lambda_k \lambda_l}{\lambda_l - \lambda_k}$ is bounded in magnitude by $\frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2}$. From the orthonormality (2.5) and the Bessel inequality, we conclude that

$$(\int_{\mathbb{H}} e^{-2\omega} |\Delta \dot{\tilde{u}}_2|^2)^{1/2} \leqslant \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} (\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega} \tilde{u}_2^2)^{1/2}.$$

If we change coordinates by writing

$$\dot{\tilde{u}} = \dot{u} \circ \Phi$$

we conclude that

$$(\int_{ABC} |\Delta \dot{u}_2|^2)^{1/2} \leqslant \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} (\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega} \tilde{u}_2^2)^{1/2}.$$

Also, $\dot{u}_2$ has mean zero on ABC by (2.8). We conclude from Corollary 1.3 that

$$\|\dot{u}_2\|_{L^\infty} \leqslant \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} (\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega} \tilde{u}_2^2)^{1/2}.$$

Pulling back to $\mathbb{H}$, and estimating $\tilde{u}_2$ in L^∞ norm, we conclude that

$$\|\dot{\tilde{u}}_2\|_{L^\infty(\mathbb{H})} \leqslant X \|\tilde{u}_2\|_{L^\infty(\mathbb{H})}$$

where X is the explicit (but somewhat messy) quantity

$$X := \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} (\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega})^{1/2}.$$

This gives stability of the second eigenfunction in L^∞ norm, as long as there is an eigenvalue gap $\lambda_3 - \lambda_2 > 0$.

One can compute one factor in the quantity X as follows. Observe that

$$\int_{\mathbb{H}} e^{2\omega} = |ABC| = \frac{1}{2\pi^2} \Gamma(\alpha/\pi)^2 \Gamma(\beta/\pi)^2 \Gamma(\gamma/\pi)^2 \sin(\alpha)\sin(\beta)\sin(\gamma).$$

If we view α, β, γ (and hence ω) as varying linearly in time, and differentiate the above equation under the integral sign twice in time, we conclude that

$$\begin{aligned} 4\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega} &= \frac{d^2}{dt^2} \int_{\mathbb{H}} e^{2\omega} \\ &= (\dot{\alpha}^2 \frac{\partial^2}{\partial \alpha^2} + 2\dot{\alpha}\dot{\beta} \frac{\partial^2}{\partial\alpha\partial\beta} + 2\dot{\alpha}\dot{\gamma}\frac{\partial^2}{\partial\alpha\partial\gamma} \\ &\quad + \dot{\beta}^2 \frac{\partial^2}{\partial\beta^2} + 2\dot{\beta}\dot{\gamma}\frac{\partial^2}{\partial\beta\gamma} + \dot{\gamma}^2) \\ &\quad\quad (\frac{1}{2\pi^2} \Gamma(\alpha/\pi)^2 \Gamma(\beta/\pi)^2 \Gamma(\gamma/\pi)^2 \sin(\alpha)\sin(\beta)\sin(\gamma)). \end{aligned}$$

This, in principle, expresses the factor $(\int_{\mathbb{H}} |\dot{\omega}|^2 e^{2\omega})^{1/2}$ in X as an explicit combination of trigonometric functions, gamma functions, and the first two derivatives of the gamma function. However, this formula is somewhat messy, to say the least.