1. A Sobolev inequality

**Lemma 1.1.** Let $ABCD$ be a parallelogram, and let $u$ be a $C^2$ function on $ABCD$. Let $\alpha$ be the angle subtended by $A$. Then

$$|u(A) - u(B) + u(C) - u(D)| \leq \frac{1}{\sin(\alpha)} \int_{ABCD} |\nabla^2 u|.$$

**Proof.** We may normalise $A = 0$, so that $C = B + D$. From two applications of the fundamental theorem of calculus one has

$$\int_0^1 \int_0^1 \partial_{st} u(sB + tD) = u(A) - u(B) + u(C) - u(D).$$

The left-hand side can be rewritten as

$$\frac{1}{|ABCD|} \int_{ABCD} (B \cdot \nabla)(D \cdot \nabla)u.$$

Since

$$|ABCD| = |B||D|\sin(\alpha)$$

the claim follows. \hfill \Box

**Lemma 1.2.** Let $ABC$ be a triangle with angles $\alpha, \beta, \gamma$, and let $u$ be a $C^2$ function on $ABC$ that obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of $ABC$. Then for any $P \in ABC$, one has

$$|u(P) - u(Q)| \leq \frac{1}{3\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.$$

**Proof.** Let $P \in ABC$. Let $D, E \in AB, F, G \in BC, H, I \in AC$ be the points such that $ADXI, BFXE, CHXG$ are parallelograms (thus $D, G$ are the intersections with $AB, BC$ respectively of the line through $X$ parallel to $AC$, and so forth. Then from the preceding lemma one has

$$|u(X) + u(A) - u(D) - u(I)| \leq \frac{1}{\sin(\alpha)} \int_{ADXI} |\nabla^2 u|$$

$$|u(X) + u(B) - u(F) - u(E)| \leq \frac{1}{\sin(\beta)} \int_{BFXE} |\nabla^2 u|$$

$$|u(X) + u(C) - u(H) - u(G)| \leq \frac{1}{\sin(\gamma)} \int_{CHXG} |\nabla^2 u|.$$
Also, by reflecting the triangles $DEX, FGX, XHI$ across the Neumann boundary and using the previous lemma, we see that

\[
|u(X) + u(X) - u(D) - u(E)| \leq \frac{2}{\sin(\gamma)} \int_{DEX} |\nabla^2 u|
\]

\[
|u(X) + u(X) - u(F) - u(G)| \leq \frac{2}{\sin(\alpha)} \int_{FGX} |\nabla^2 u|
\]

\[
|u(X) + u(X) - u(H) - u(I)| \leq \frac{2}{\sin(\beta)} \int_{XHI} |\nabla^2 u|.
\]

Summing the latter three combinations of $u$ and subtracting the former three using the triangle inequality, we conclude that

\[
|3u(X) - u(A) - u(B) - u(C)| \leq \frac{2}{\min(\sin(\alpha), \sin(\beta), \sin(\gamma))} \int_{ABC} |\nabla^2 u|.
\]

A similar inequality holds with $X$ replaced by $Y$. Subtracting the two inequalities, we obtain the claim.

**Corollary 1.3.** Let $ABC$ be a triangle with angles $\alpha, \beta, \gamma$, and let $u$ be a $C^2$ function on $ABC$ which is smooth up to the boundary except possibly at the vertices $A, B, C$, and which obeys the Neumann boundary condition $n \cdot \nabla u = 0$ on the boundary of $ABC$, and has mean zero on $ABC$. Then

\[
\|u\|_{L^\infty(ABC)} \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \|\Delta u\|_{L^2(ABC)}.
\]

**Proof.** If $u$ has mean zero, then $\|u\|_{L^\infty(ABC)}$ is bounded by $|u(P) - u(Q)|$ for some $P, Q \in ABC$. From the previous lemma we thus have

\[
\|u\|_{L^\infty(ABC)} \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^{1/2} \|\nabla^2 u\|_{L^2(ABC)}.
\]

It will thus suffice to show the Bochner-Weitzenbock identity

\[
\int_{ABC} |\nabla^2 u|^2 = \int_{ABC} |\Delta u|^2.
\]

But this can be accomplished by two integration by parts, using the smoothness and Neumann boundary hypotheses on $u$ (and a regularisation argument if necessary to cut away from the vertices) *more details needed here.*

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2. **Schwarz-Christoffel**

Let $0 < \alpha, \beta, \gamma < \pi$ be angles adding up to $\pi$, then we can define a Schwarz-Christoffel map $\Phi_{\alpha,\beta} : \mathbb{H} \to ABC$ from the half-plane $\mathbb{H} := \{z : \Im(z) > 0\}$ to a triangle $ABC$ with angles $\alpha, \beta, \gamma$ by the formula

\[
\Phi_{\alpha,\beta}(z) := \int_0^z \frac{d\zeta}{\zeta^{1-\alpha/\pi} (1 - \zeta)^{1-\beta/\pi}}.
\]
where the integral is over any contour from 0 to \( z \) in \( \mathbb{H} \), and one chooses the branch cut to make both factors in the denominator positive real on the interval \([0, 1]\). Thus the vertices of the triangle are given by

\[
A := \Phi_{\alpha, \beta}(0) = 0
\]

\[
B := \Phi_{\alpha, \beta}(1) = \int_0^1 \frac{dt}{t^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}} = \frac{\Gamma((\alpha/\pi)\Gamma((\beta/\pi))}{\Gamma((\alpha + \beta)/\pi)}
\]

\[
C := \Phi_{\alpha, \beta}(\infty) = -e^{i\alpha} \int_{-\infty}^0 \frac{dt}{|t|^{1-\alpha/\pi}(1-t)^{1-\beta/\pi}}
\]

\[
= e^{i\alpha} \int_1^\infty \frac{ds}{(s-1)^{1-\alpha/\pi}s^{1-\beta/\pi}}
\]

\[
= e^{i\alpha} \int_0^1 \frac{dv}{(v-1)^{1-\alpha/\pi}v^{1-\gamma/\pi}}
\]

\[
= e^{i\alpha} \frac{\Gamma((\alpha/\pi)\Gamma((\gamma/\pi))}{\Gamma((\alpha + \gamma)/\pi)}
\]

where we have used the beta function identity

\[
\int_0^1 \frac{dt}{t^{1-x}(1-t)^{1-y}} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}
\]

and the changes of variable \( s = 1 - t, v = 1/s \). In particular, the area of the triangle \( ABC \) can be expressed as

\[
|ABC| = \frac{1}{2} |B||C| \sin(\alpha) = \frac{\Gamma((\alpha/\pi)\Gamma((\beta/\pi)\Gamma((\gamma/\pi))}{2\Gamma((\alpha + \beta)/\pi)\Gamma((\alpha + \gamma)/\pi)} \sin(\alpha)
\]

which can be simplified using the formula \( \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin(\pi z)} \) as the more symmetric expression

\[
|ABC| = \frac{1}{2\pi^2} \Gamma((\alpha/\pi)^2\Gamma((\beta/\pi)\Gamma((\gamma/\pi)^2 \sin(\alpha) \sin(\beta) \sin(\gamma)). \quad (2.1)
\]

We write

\[
|\Phi'_{\alpha, \beta}(z)| = e^{\omega(z)}
\]

where \( \omega = \omega_{\alpha, \beta} \) is the harmonic function

\[
\omega(z) := \left( \frac{\alpha}{\pi} - 1 \right) \log |z| + \left( \frac{\beta}{\pi} - 1 \right) \log |1 - z|. \quad (2.2)
\]

If \( u : ABC \to \mathbb{R} \) is a smooth function, and \( \tilde{u} : \mathbb{H} \to \mathbb{R} \) is its pullback to the half-plane \( \mathbb{H} \) defined by

\[
\tilde{u} := u \circ \Phi_{\alpha, \beta}
\]

then we have

\[
\int_{ABC} u = \int_{\mathbb{H}} e^{2\omega} \tilde{u}.
\]
In a similar vein we have the conformal invariance of the two-dimensional Dirichlet energy

\[ \int_{ABC} |\nabla u|^2 = \int_{\mathbb{H}} |\nabla \tilde{u}|^2 \]

and the conformal transformation of the Laplacian:

\[ \Delta \tilde{u}(z) = e^{2\omega} \Delta u. \]

In particular, the Rayleigh quotient

\[ \frac{\int_{ABC} |\nabla u|^2}{\int_{ABC} |u|^2} \]

with mean zero condition \( \int_{ABC} u = 0 \) becomes, when pulled back to \( \mathbb{H} \), the Rayleigh quotient

\[ \frac{\int_{\mathbb{H}} |\nabla \tilde{u}|^2}{\int_{\mathbb{H}} e^{2\omega} |\tilde{u}|^2} \]

with mean zero condition \( \int_{\mathbb{H}} e^{2\omega} \tilde{u} = 0 \).

Let \( u_2, u_3, \ldots \) be an \( L^2 \)-normalised eigenbasis for the Neumann Laplacian \( -\Delta \) on \( ABC \) with eigenvalues \( \lambda_2 \leq \lambda_3 \leq \ldots \), thus

\[ -\Delta u_k = \lambda_k u_k \]
on \( ABC \) with Neumann boundary data

\[ n \cdot \nabla u_k = 0 \]

and orthonormality

\[ \int_{ABC} u_j u_k = \delta_{jk} \]

and mean zero condition

\[ \int_{ABC} u_j = 0. \]

One can show that when \( ABC \) is acute-angled, these eigenfunctions are smooth except possibly at the vertices \( A, B, C \), and are uniformly \( C^2 \). \textbf{add details here}

Pulling all this back to \( \mathbb{H} \), we obtain transformed eigenfunctions \( \tilde{u}_2, \tilde{u}_3, \ldots \) on \( \mathbb{H} \) to the conformal eigenfunction equation

\[ -\Delta \tilde{u}_k = \lambda_k e^{2\omega} \tilde{u}_k \]  \hspace{1cm} (2.3)
on \( \mathbb{H} \) with Neumann boundary data

\[ n \cdot \nabla \tilde{u}_k = 0 \]  \hspace{1cm} (2.4)
and orthonormality

\[ \int_{\mathbb{H}} e^{2\omega} \tilde{u}_j \tilde{u}_k = \delta_{jk} \]  \hspace{1cm} (2.5)
and mean zero condition

\[ \int_{\mathbb{H}} e^{2\omega} \tilde{u}_j = 0. \]  \hspace{1cm} (2.6)
Now suppose that we vary the angle parameters $\alpha, \beta, \gamma$ smoothly with respect to some time parameter $t$, thus also varying the triangles $ABC$, eigenfunctions $u_k$ and transformed eigenfunctions $\tilde{u}_k$, eigenvalues $\lambda_k$, and conformal factor $\omega$. We will use dots to indicate time differentiation, thus for instance $\dot{\alpha} = \frac{d}{dt}\alpha$. Let us formally suppose that all of the above data vary smoothly (or at least $C^1$) in time we will eventually need to justify this, of course. Since $\alpha + \beta + \gamma = \pi$, we have

$$\dot{\alpha} + \dot{\beta} + \dot{\gamma} = 0.$$  

The variation $\dot{\omega}$ of the conformal factor is explicitly computable from (2.2) as being a logarithmic weight:

$$\dot{\omega} = \frac{\dot{\alpha}}{\pi} \log |z| + \frac{\dot{\beta}}{\pi} \log |1 - z|.$$  

Next, by (formally) differentiating (2.3) we obtain an equation for the variation $\dot{\tilde{u}}_k$ of the $k^{th}$ eigenfunction:

$$-\Delta \dot{\tilde{u}}_k = \lambda_k e^{2\omega} \tilde{u}_k + 2\lambda_k \omega e^{2\omega} \tilde{u}_k + \lambda_k e^{2\omega} \dot{\tilde{u}}_k.$$  

To solve this equation for $\dot{\tilde{u}}_k$, we observe from differentiating (2.4), (2.5), (2.6) that

$$\int_H e^{2\omega} \dot{\tilde{u}}_k = 0$$  

and

$$\int_H e^{2\omega} \tilde{u}_k \dot{\tilde{u}}_k = 0$$  

and

$$n \cdot \nabla \dot{\tilde{u}}_k = 0.$$  

By eigenfunction expansion, we thus have

$$\dot{\tilde{u}}_k = \sum_{l \neq k} (\int_H e^{2\omega} \tilde{u}_l \dot{\tilde{u}}_k) \tilde{u}_l$$  

in a suitable sense ($L^2$ with weight $e^{\omega}$). Now we evaluate the expression in parentheses. Integrating (2.7) against $\tilde{u}_l$ and using (2.5) reveals that

$$-\int_H \Delta \dot{\tilde{u}}_k \tilde{u}_l = 2\lambda_k \int_H \omega e^{2\omega} \tilde{u}_k \tilde{u}_l + \lambda_k \int_H e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l.$$  

By Green’s theorem and the Neumann conditions on $\dot{\tilde{u}}_k$ and $\tilde{u}_l$, the left-hand side is

$$- \int_H \dot{\tilde{u}}_k \Delta \tilde{u}_l$$  

which by (2.3) is equal to

$$\lambda_l \int_H e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l.$$  

Inserting this into (2.10) we see that

$$\int_H e^{2\omega} \dot{\tilde{u}}_k \tilde{u}_l = \frac{2\lambda_k}{\lambda_l - \lambda_k} \int_H \omega e^{2\omega} \tilde{u}_k \tilde{u}_l$$.
and thus by (2.9)
\[ \hat{u}_k = \sum_{l \neq k} \left( \frac{2\lambda_k}{\lambda_l - \lambda_k} \int_H \omega e^{2\omega} \hat{u}_k \hat{u}_l \right) \hat{u}_l. \] (2.11)

We can take Laplacians and conclude that
\[ -\Delta \hat{u}_k = e^{2\omega} \sum_{l \neq k} \left( \frac{2\lambda_k\lambda_l}{\lambda_l - \lambda_k} \int_H \omega e^{2\omega} \hat{u}_k \hat{u}_l \right) \hat{u}_l. \]

Set \( k = 2 \), then \( \frac{2\lambda_k\lambda_l}{\lambda_l - \lambda_k} \) is bounded in magnitude by \( \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} \). From the orthonormality (2.5) and the Bessel inequality, we conclude that
\[ \left( \int_H e^{-2\omega} |\Delta \hat{u}_2|^2 \right)^{1/2} \leq \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} \left( \int_H |\hat{\omega}|^2 e^{2\omega} \hat{u}_2^2 \right)^{1/2}. \]
If we change coordinates by writing
\[ \hat{u} = \hat{u} \circ \Phi \]
we conclude that
\[ \left( \int_{ABC} |\Delta \hat{u}_2|^2 \right)^{1/2} \leq \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} \left( \int_H |\hat{\omega}|^2 e^{2\omega} \hat{u}_2^2 \right)^{1/2}. \]

Also, \( \hat{u}_2 \) has mean zero on \( ABC \) by (2.8). We conclude from Corollary 1.3 that
\[ \|\hat{u}_2\|_{L^\infty} \leq \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^1/2 \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} \left( \int_H |\hat{\omega}|^2 e^{2\omega} \hat{u}_2^2 \right)^{1/2}. \]
Pulling back to \( H \), and estimating \( \hat{u}_2 \) in \( L^\infty \) norm, we conclude that
\[ \|\hat{u}_2\|_{L^\infty(H)} \leq X \|\hat{u}_2\|_{L^\infty(H)} \]
where \( X \) is the explicit (but somewhat messy) quantity
\[ X := \frac{4}{3 \min(\sin(\alpha), \sin(\beta), \sin(\gamma))} |ABC|^1/2 \frac{2\lambda_2\lambda_3}{\lambda_3 - \lambda_2} \left( \int_H |\hat{\omega}|^2 e^{2\omega} \right)^{1/2}. \]
This gives stability of the second eigenfunction in \( L^\infty \) norm, as long as there is an eigenvalue gap \( \lambda_3 - \lambda_2 > 0 \).