

ω -Stable Theories: Introduction

The main references for these notes are David Marker's book "Model Theory: An Introduction" and Matthias Aschenbrenner's model theory course taught at UCLA in Spring 2009.

1 ω - Stable/Totally Transcendental Theories

Throughout let T be a complete theory in a countable language \mathcal{L} having infinite models.

For an \mathcal{L} -structure \mathcal{M} and $A \subseteq M$ let $S_n^{\mathcal{M}}(A)$ denote the set of n -types of A . We define a topology (called Stone topology) on $S_n^{\mathcal{M}}(A)$ by setting basic open sets to be of the form $U_\phi = \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$ where ϕ is an $\mathcal{L}(A)$ -formula. Then $S_n^{\mathcal{M}}(A)$ is totally disconnected, Hausdorff and compact (the latter follows from the compactness theorem).

Definition 1.1. *Let κ be an infinite cardinal. We say that T is κ -stable if whenever $\mathcal{M} \models T$, $A \subseteq M$, and $|A| \leq \kappa$, then $|S_n^{\mathcal{M}}(A)| \leq \kappa$ for all $n \in \mathbb{N}$.*

Examples:

- ACF, the theory of algebraically closed fields is λ -stable for all $\lambda \geq \omega$ for the following reason. Let $K \models \text{ACF}$ and $F \subset K$ be a subfield of cardinality λ . Using the fact that ACF admits quantifier elimination, one can show that the map $S_n^K(F) \rightarrow \text{Spec}(F[\bar{x}])$ by $p \mapsto \{f(\bar{x}) \in F[\bar{x}] : "f(\bar{x}) = 0" \in p\}$ is an injection (it is actually a continuous bijection if we equip $S_n^K(F)$ and $\text{Spec}(F[\bar{x}])$ with Stone and Zariski topologies, respectively). By Hilbert's Basis Theorem, all ideals in $F[\bar{x}]$ are finitely generated. Thus $|S_n^K(F)| \leq |\text{Spec}(F[\bar{x}])| \leq |F| + \aleph_0$.
- A theory T is called κ -categorical if any two models of T of cardinality κ are isomorphic. Morley's theorem states that if a countable complete

theory is κ -categorical for some uncountable κ then it is λ -categorical for every uncountable λ .

Any uncountably categorical theory T is ω -stable (the next example shows that ω -categorical theories need not be ω -stable). To see why T is ω -stable let T be κ -categorical for some $\kappa > \aleph_0$. Using order indiscernibles, one can construct a model of T of cardinality κ that realizes only countably many types over every countable subset. But assuming that T is not ω -stable and taking a transfinite chain of elementary extensions we can get a model of cardinality κ realizing uncountably many types over some countable set. Thus the two constructed models are not isomorphic, contradicting the κ -categoricity of T .

- DLO, the theory of dense linear orderings without endpoints, is NOT ω -stable (although it is ω -categorical): there is a one-to-one correspondence between Dedekind cuts and types in $S_1^{\mathbb{Q}}(\mathbb{Q})$. Hence $|S_1^{\mathbb{Q}}(\mathbb{Q})| = 2^{\aleph_0}$.

Definition 1.2. Let $\mathcal{M} \models T$. A binary tree of $\mathcal{L}(M)$ -formulas is a collection of $\mathcal{L}(M)$ -formulas $\{\phi_s(\bar{x}) : s \in 2^{<\omega}\}$ that is such that for all $s \in 2^{<\omega}$:

- $T \cup \{\phi_s(\bar{x})\}$ is consistent;
- $T \models \forall \bar{x}(\phi_{s \frown i}(\bar{x}) \rightarrow \phi_s(\bar{x}))$, for $i = 0, 1$;
- $T \models \forall \bar{x} \neg(\phi_{s \frown 0}(\bar{x}) \wedge \phi_{s \frown 1}(\bar{x}))$.

Definition 1.3. A theory T is called totally transcendental (t.t.) if there is no binary tree of $\mathcal{L}(M)$ -formulas for any $\mathcal{M} \models T$.

Theorem 1.4. TFAE:

- (i) T is t.t.;
- (ii) T is λ -stable for every $\lambda \geq \aleph_0$;
- (iii) T is ω -stable.

Proof.

(i) \Rightarrow (ii): Assume T is not λ -stable for some $\lambda \geq \aleph_0$, i.e. $|S_n^{\mathcal{M}}(A)| > \lambda$ for some $n \in \mathbb{N}$, $A \subseteq M$, $|A| \leq \lambda$, $\mathcal{M} \models T$. For an $L(A)$ -formula $\phi(\bar{x})$ let $U_\phi = \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$ (basic open set in the Stone topology). Call

an $L(A)$ -formula $\phi(\bar{x})$ large if $|U_\phi| > \lambda$ (otherwise call it small). Since there are only λ many $L(A)$ -formulas, there is a large formula $\phi(\bar{x})$ (e.g. $\phi(\bar{x}) = \bar{x} = \bar{x}$). Because $|U_\phi \setminus \bigcup_{\psi(\bar{x}) \text{ small}} U_\psi| > \lambda$, there exists $p, q \in U_\phi$, $p \neq q$,

consisting entirely of large formulas. Take $\psi(\bar{x}) \in p$ such that $\neg\psi(\bar{x}) \in q$. Then both $\phi(\bar{x}) \wedge \psi(\bar{x})$ and $\phi(\bar{x}) \wedge \neg\psi(\bar{x})$ are large. Continuing in this manner we construct a binary tree of $\mathcal{L}(A)$ -formulas, and hence T is not t.t.

(ii) \Rightarrow (iii): Trivial.

(iii) \Rightarrow (i): Assume there is a binary tree S of $\mathcal{L}(M)$ -formulas for some $\mathcal{M} \models T$, and let $A \subseteq M$ be the countable set of parameters that the formulas in S use. Then the set of formulas in each infinite branch of S forms a consistent partial type over A the completion of which to a type over A is different for different branches. Hence $|S_n^{\mathcal{M}}(A)| \geq 2^{\aleph_0}$ and thus T is not ω -stable. \square

The example of DLO suggests that if T is ω -stable then T doesn't have a model with a definable ordering on it. In fact, something stronger is true as the following theorem shows.

Definition 1.5. *We say that T has the order property if there is an \mathcal{L} -formula $\phi(\bar{v}, \bar{w})$ and $\mathcal{M} \models T$ with $X = \{\bar{x}_1, \bar{x}_2, \dots\} \subseteq M$ such that ϕ defines a linear order on X , i.e. $\mathcal{M} \models \phi(\bar{x}_i, \bar{x}_j) \Leftrightarrow i < j$.*

Sometimes the order property is used in a slightly different form provided by the following lemma.

Lemma 1.6. *T has the order property if and only if there is an \mathcal{L} -formula $\phi(\bar{v}, \bar{w})$ and $\mathcal{M} \models T$ with $X = \{\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2, \dots\} \subseteq M$ such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$.*

Proof.

\Rightarrow : Having $X = \{\bar{a}_1, \bar{a}_2, \dots\} \subseteq M$ with a definable order, set $\bar{b}_i = \bar{a}_i$.

\Leftarrow : Having $X = \{\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2, \dots\} \subseteq M$ with an \mathcal{L} -formula $\phi(\bar{v}, \bar{w})$ such that $\mathcal{M} \models \phi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i < j$, set $\bar{c}_i = (\bar{a}_i, \bar{b}_i)$ and let $\psi(\bar{v}_1, \bar{w}_1, \bar{v}_2, \bar{w}_2) = \phi(\bar{v}_1, \bar{w}_2)$. Then ψ defines an order on $\{\bar{c}_1, \bar{c}_2, \dots\}$. \square

Theorem 1.7 (No order property). *If T is κ -stable for some $\kappa \geq \aleph_0$, then T doesn't have the order property.*

To prove this we first need the following lemma.

Lemma 1.8. *For any $\kappa \geq \aleph_0$, there is a dense linear order $(A, <)$ with $B \subset A$ such that B is dense in A and $|B| \leq \kappa < |A|$.*

Proof. Let λ be the least such that $2^\lambda > \kappa$. Take $A = \mathbb{Q}^\lambda$, the set of all functions from λ to \mathbb{Q} and order A lexicographically, i.e. $f < g$ if and only if $f(\alpha) < g(\alpha)$, where α is the least such that $f(\alpha) \neq g(\alpha)$. Clearly, $(A, <) \models$ DLO. Let $B \subset A$ be the set of λ -sequences that are eventually 0. Then clearly B is dense in A and $|B| = \sup_{\mu < \lambda} 2^\mu \leq \kappa$. \square

Proof of Theorem 1.7. Assume for contradiction that there is a formula $\phi(\bar{v}, \bar{w})$ and $\mathcal{M} \models T$ with $X = \{\bar{x}_1, \bar{x}_2, \dots\} \subseteq M$ such that ϕ defines a linear order on X . Let $(A, <)$ and $B \subset A$ be as in Lemma 1.8. Then, using the compactness theorem, we can pass to an elementary extension \mathcal{N} of \mathcal{M} containing a sequence Y of order type A whose order is defined by ϕ . Let Y_0 be the subset of Y corresponding to B .

For any $\bar{x} < \bar{y} \in Y$ there is $\bar{z} \in Y_0$ such that $\bar{x} < \bar{z} < \bar{y}$. Then $\mathcal{N} \models \phi(\bar{x}, \bar{z})$, but $\mathcal{N} \models \neg\phi(\bar{y}, \bar{z})$. Thus any two distinct elements of Y realize distinct 1-types over Y_0 . Because $|Y_0| \leq \kappa < |Y|$, T is not κ -stable, a contradiction. \square

Using Ramsey's theorem one can always find a model of any countable theory containing an infinite sequence of order indiscernibles. But not all theories have models with infinite sets of indiscernibles (tuples satisfy the same types regardless the order). In case of κ -stable theories, the fact that T doesn't have the order property (or rather the proof of this fact) gives us the following theorem.

Theorem 1.9. *Let T be a κ -stable for some $\kappa \geq \aleph_0$. If $\mathcal{M} \models T$ and $X \subseteq M$ is an infinite sequence of order indiscernibles, then X is a set of indiscernibles.*

Proof. Let $\phi(\bar{v})$ be an \mathcal{L} -formula and $\bar{x} = (x_1, \dots, x_n)$ be an increasing sequence in X such that $\mathcal{M} \models \phi(x_1, \dots, x_n)$. Because any permutation is a product of transpositions, it is enough to prove that

$$\mathcal{M} \models \phi(x_1, \dots, x_{m-1}, x_{m+1}, x_m, x_{m+2}, \dots, x_n).$$

Assume for contradiction that $\mathcal{M} \models \neg\phi(x_1, \dots, x_{m-1}, x_{m+1}, x_m, x_{m+2}, \dots, x_n)$.

Let $(A, <)$ and $B \subset A$ be as in Lemma 1.8. We can find $\mathcal{N} \models T$ containing a sequence of order indiscernibles of order type $(A, <)$ with $\text{tp}(X) = \text{tp}(Y)$ (if

$X \subseteq M$ is a sequence of order indiscernibles of order type $(I, <)$ then $\text{tp}(X) = \{\psi(v_1, \dots, v_n) : \mathcal{M} \models \psi(x_{i_1}, \dots, x_{i_n}), x_{i_1}, \dots, x_{i_n} \in X, i_1 < \dots < i_n \in I, n \in \mathbb{N}\}$. Let Y_0 be the subset of Y corresponding to B . If $y_1 < \dots < y_n \in Y$ then $\mathcal{N} \models \phi(y_1, \dots, y_n)$ and $\mathcal{N} \models \neg\phi(y_1, \dots, y_{m-1}, y_{m+1}, y_m, y_{m+2}, \dots, y_n)$.

For any $x < y \in Y$ there are $z_1, \dots, z_{n-1} \in Y_0$ such that $z_1 < \dots < z_{m-1} < x < z_m < y < z_{m+1} < \dots < z_{n-1}$. Then

$$\mathcal{N} \models \phi(z_1, \dots, z_{m-1}, x, z_m, z_{m+2}, \dots, z_{n-1}),$$

but

$$\mathcal{N} \models \neg\phi(z_1, \dots, z_{m-1}, y, z_m, z_{m+2}, \dots, z_{n-1}).$$

Thus any two distinct elements of Y realize distinct 1-types over Y_0 . Because $|Y_0| \leq \kappa < |Y|$, T is not κ -stable, a contradiction. \square

2 Morley Rank

Morley rank provides a notion of “dimension” of definable or type definable sets and is apparently one of the most important tools for analyzing ω -stable theories. To get motivated let’s consider the notion of dimension in linear algebra. Suppose that K is an infinite field, $V \subseteq K^n$ is an m -dimensional vector space, and f is a non-zero linear functional on V . For $a \in K$, let $V_a = \{x \in V : f(x) = a\}$. Then $\{V_a : a \in K\}$ is an infinite family of $(m-1)$ -dimensional affine subsets of V . Morley rank is an attempt to generalize this property of dimension. The basic idea is that if a definable set X contains infinitely many pairwise disjoint sets of dimension m , then X should have dimension at least $m+1$.

Definition 2.1. *Let \mathcal{M} be an \mathcal{L} -structure and $\phi(\bar{v})$ be an $\mathcal{L}(M)$ -formula. We define $\text{MR}^{\mathcal{M}}(\phi)$, the Morley rank of ϕ in \mathcal{M} , as follows. Inductively define $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha$ for an ordinal α :*

- (i) $\text{MR}^{\mathcal{M}}(\phi) \geq 0$ if and only if $\phi(\mathcal{M}) \neq \emptyset$;
- (ii) if α is a limit ordinal, then $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha$ if and only if $\text{MR}^{\mathcal{M}}(\phi) \geq \beta$ for all $\beta < \alpha$;
- (iii) if $\alpha = \beta + 1$, then $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha$ if and only if there are $\mathcal{L}(M)$ -formulas $\psi_1(\bar{v}), \psi_2(\bar{v}), \dots$ such that $\psi_1(\mathcal{M}), \psi_2(\mathcal{M}), \dots$ is an infinite family of pairwise disjoint subsets of $\phi(\mathcal{M})$ and $\text{MR}^{\mathcal{M}}(\psi_i) \geq \beta$ for all i .

If $\phi(\mathcal{M}) = \emptyset$ then $\text{MR}^{\mathcal{M}}(\phi) = -1$. If $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha$ but $\text{MR}^{\mathcal{M}}(\phi) \not\geq \alpha+1$, then $\text{MR}^{\mathcal{M}}(\phi) = \alpha$. If $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha$ for all ordinals α , then $\text{MR}^{\mathcal{M}}(\phi) = \infty$.

Since we usually work within a monster model, we would like to modify this definition so that it doesn't really depend on the model containing the parameters that ϕ uses. The next lemmas show how to eliminate that dependence.

Lemma 2.2. *Let $\phi(\bar{v}, \bar{w})$ be an \mathcal{L} -formula, \mathcal{M} be an \aleph_0 -saturated structure and $\bar{a}, \bar{b} \in M$. If $\text{tp}^{\mathcal{M}}(\bar{a}) = \text{tp}^{\mathcal{M}}(\bar{b})$ then $\text{MR}^{\mathcal{M}}(\phi(\bar{v}, \bar{a})) = \text{MR}^{\mathcal{M}}(\phi(\bar{v}, \bar{b}))$.*

Proof. Assuming $\text{tp}^{\mathcal{M}}(\bar{a}) = \text{tp}^{\mathcal{M}}(\bar{b})$ we prove by induction on α that $\text{MR}^{\mathcal{M}}(\phi(\bar{v}, \bar{a})) \geq \alpha$ if and only if $\text{MR}^{\mathcal{M}}(\phi(\bar{v}, \bar{b})) \geq \alpha$.

For $\alpha = 0$, $\phi(\mathcal{M}, \bar{a}) = \emptyset$ if and only if $\neg\exists\bar{v}\phi(\bar{v}, \bar{w}) \in \text{tp}^{\mathcal{M}}(\bar{a})$ if and only if $\neg\exists\bar{v}\phi(\bar{v}, \bar{w}) \in \text{tp}^{\mathcal{M}}(\bar{b})$ if and only if $\phi(\mathcal{M}, \bar{b}) = \emptyset$.

The case when α is a limit ordinal is clear. So, suppose the claim is true for α and $\text{MR}^{\mathcal{M}}(\phi(\bar{v}, \bar{a})) \geq \alpha+1$. Then there are \mathcal{L} -formulas $\psi_1(\bar{v}, \bar{w}_1), \psi_2(\bar{v}, \bar{w}_2), \dots, \bar{c}_1, \bar{c}_2, \dots \in M$ such that $\psi_1(\mathcal{M}, \bar{c}_1), \psi_2(\mathcal{M}, \bar{c}_2), \dots$ is an infinite family of pairwise disjoint subsets of $\phi(\mathcal{M})$ and $\text{MR}^{\mathcal{M}}(\psi_i(\bar{v}, \bar{c}_i)) \geq \alpha$ for all i . Because \mathcal{M} is \aleph_0 -saturated, we can inductively find $\bar{d}_1, \bar{d}_2, \dots \in M$ such that $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{c}_1, \dots, \bar{c}_m) = \text{tp}^{\mathcal{M}}(\bar{b}, \bar{d}_1, \dots, \bar{d}_m)$ for all $m < \omega$. Then $\psi_1(\mathcal{M}, \bar{d}_1), \psi_2(\mathcal{M}, \bar{d}_2), \dots$ is an infinite sequence of pairwise disjoint subsets of $\phi(\mathcal{M}, \bar{b})$, and, by induction, $\text{MR}^{\mathcal{M}}(\psi_i(\bar{v}, \bar{d}_i)) \geq \alpha$. Thus $\text{MR}^{\mathcal{M}}(\phi(\bar{v}, \bar{b})) \geq \alpha + 1$. \square

Lemma 2.3. *If \mathcal{M} and \mathcal{N} are \aleph_0 -saturated structures and $\mathcal{M} \prec \mathcal{N}$, then for any $\mathcal{L}(\mathcal{M})$ -formula ϕ , $\text{MR}^{\mathcal{M}}(\phi) = \text{MR}^{\mathcal{N}}(\phi)$.*

Proof. We prove by induction on α that $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha$ if and only if $\text{MR}^{\mathcal{N}}(\phi) \geq \alpha$.

The case $\alpha = 0$ follows from elementarity and the case when α is a limit is straightforward. Also, if $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha + 1$ then it follows directly from the definition of Morley rank (using elementarity and the induction hypothesis) that $\text{MR}^{\mathcal{N}}(\phi) \geq \alpha + 1$.

So, assume $\text{MR}^{\mathcal{N}}(\phi) \geq \alpha+1$, i.e. there are \mathcal{L} -formulas $\psi_1(\bar{v}, \bar{w}_1), \psi_2(\bar{v}, \bar{w}_2), \dots, \bar{c}_1, \bar{c}_2, \dots \in N$ such that $\psi_1(\mathcal{N}, \bar{c}_1), \psi_2(\mathcal{N}, \bar{c}_2), \dots$ is an infinite family of pairwise disjoint subsets of $\phi(\mathcal{N})$ and $\text{MR}^{\mathcal{N}}(\psi_i(\bar{v}, \bar{c}_i)) \geq \alpha$ for all i . Let $\psi(\bar{v}, \bar{w})$ be an \mathcal{L} -formula such that $\phi(\bar{v}) = \psi(\bar{v}, \bar{a})$ for some $\bar{a} \in M$. By elementarity, $\text{tp}^{\mathcal{N}}(\bar{a}) = \text{tp}^{\mathcal{M}}(\bar{a})$ and hence, because \mathcal{M} is \aleph_0 -saturated, we can inductively find $\bar{d}_1, \bar{d}_2, \dots \in M$ such that $\text{tp}^{\mathcal{N}}(\bar{a}, \bar{c}_1, \dots, \bar{c}_m) = \text{tp}^{\mathcal{M}}(\bar{a}, \bar{d}_1, \dots, \bar{d}_m)$ for all $m < \omega$. By elementarity, $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{d}_1, \dots, \bar{d}_m) = \text{tp}^{\mathcal{N}}(\bar{a}, \bar{d}_1, \dots, \bar{d}_m)$, and hence

$\text{tp}^{\mathcal{N}}(\bar{a}, \bar{c}_1, \dots, \bar{c}_m) = \text{tp}^{\mathcal{N}}(\bar{a}, \bar{d}_1, \dots, \bar{d}_m)$ for all $m < \omega$. Thus, by Lemma 2.2, $\text{MR}^{\mathcal{N}}(\psi_i(\bar{v}, \bar{d}_i)) \geq \alpha$ for all i . By induction, $\text{MR}^{\mathcal{M}}(\psi_i(\bar{v}, \bar{d}_i)) \geq \alpha$ for all i and consequently, $\text{MR}^{\mathcal{M}}(\phi) \geq \alpha + 1$. \square

Corollary 2.4. *If \mathcal{M} is an \mathcal{L} -structure, \mathcal{N}_0 and \mathcal{N}_1 are \aleph_0 -saturated elementary extensions of \mathcal{M} , ϕ is an $\mathcal{L}(M)$ -formula, then $\text{MR}^{\mathcal{N}_0}(\phi) = \text{MR}^{\mathcal{N}_1}(\phi)$.*

Proof. Let \mathcal{N} be an \aleph_0 -saturated common elementary extension of \mathcal{N}_0 and \mathcal{N}_1 . Then by Lemma 2.3, $\text{MR}^{\mathcal{N}_0}(\phi) = \text{MR}^{\mathcal{N}}(\phi) = \text{MR}^{\mathcal{N}_1}(\phi)$. \square

Now we are ready to give a definition of Morley rank that doesn't really depend on the model containing the parameters.

Definition 2.5. *For an \mathcal{L} -structure \mathcal{M} and an \mathcal{L} -formula ϕ , define $\text{MR}(\phi)$, the Morley rank of ϕ , to be $\text{MR}^{\mathcal{N}}(\phi)$, where \mathcal{N} is any \aleph_0 -saturated elementary extension of \mathcal{M} . Also, if $X \subseteq M^n$ is defined by $\phi(\bar{v})$ then we define $\text{MR}(X)$, the Morley rank of X , to be $\text{MR}(\phi(\bar{v}))$.*

The next lemma shows that Morley rank has some properties that we would want a good notion of dimension to have.

Lemma 2.6. *Let \mathcal{M} be an \mathcal{L} -structure and let X and Y be M -definable subsets of M^n .*

- (i) *If $X \subseteq Y$, then $\text{MR}(X) \leq \text{MR}(Y)$.*
- (ii) *$\text{MR}(X \cup Y) = \max\{\text{MR}(X), \text{MR}(Y)\}$.*
- (iii) *If $X \neq \emptyset$, then $\text{MR}(X) = 0$ if and only if X is finite.*

Proof. Straightforward from the definition of Morley rank. For part (ii) show by induction that if $\text{MR}(X \cup Y) \geq \alpha$ then $\max\{\text{MR}(X), \text{MR}(Y)\} \geq \alpha$. \square

3 Monster Model

From now on we work in a monster model \mathbb{M} to avoid the awkwardness of the definition of Morley rank. We make the following assumptions on \mathbb{M} :

- \mathbb{M} is a large saturated model of T ;
- all $\mathcal{M} \models T$ that we consider are elementary submodels of \mathbb{M} with $|M| < |\mathbb{M}|$;

- we write $\text{tp}(\bar{a}/A)$ for $\text{tp}^{\mathbb{M}}(\bar{a}/A)$ and $S_n(A)$ for $S_n^{\mathbb{M}}(A)$.

The saturation of \mathbb{M} has several useful consequences.

Property 3.1 ($|\mathbb{M}|$ -compactness).

- (a) If $(X_i : i \in I)$ is a sequence of \mathbb{M} -definable subsets of \mathbb{M}^m such that $|I| < |\mathbb{M}|$ and $\mathbb{M} = \bigcup_{i \in I} X_i$, then there exists a finite $I_0 \subseteq I$ such that

$$\mathbb{M} = \bigcup_{i \in I_0} \phi_i(\mathbb{M}).$$

- (b) If $(X_i : i \in I)$ and $(Y_j : j \in J)$ are sequences of \mathbb{M} -definable subsets of \mathbb{M}^m such that $|I|, |J| < |\mathbb{M}|$ and $\bigcap_{i \in I} X_i = \bigcup_{j \in J} Y_j$, then there exist a

$$\text{finite } J_0 \subseteq J \text{ such that } \bigcap_{i \in I} X_i = \bigcup_{j \in J_0} Y_j.$$

Proof. We prove only part (a) since part (b) follows from it immediately. Because \mathbb{M} is saturated, any consistent partial type (i.e. family of closed sets with finite intersection property) over a set of cardinality smaller than $|\mathbb{M}|$ is realized in \mathbb{M} (i.e. has a nonempty intersection). But this is equivalent to the statement of part (a). \square

Property 3.2. Let X be an \mathbb{M} -definable set and $A \subset \mathbb{M}$ with $|A| < |\mathbb{M}|$. Then X is A -definable if and only if every automorphism of \mathbb{M} that fixes A pointwise fixes X setwise.

Proof. \Rightarrow : Clear.

\Leftarrow : Let $\phi(\bar{v})$ be an \mathbb{M} -formula defining X and let $\Gamma(\bar{v}, \bar{w}) = \{\phi(\bar{v}), \neg\phi(\bar{w})\} \cup \{\psi(\bar{v}) \leftrightarrow \psi(\bar{w}) : \psi(\bar{x}) \text{ is an } \mathcal{L}(A)\text{-formula}\} \cup \text{Th}(\mathbb{M}, \mathcal{L}(\mathbb{M}))$, where $\text{Th}(\mathbb{M}, \mathcal{L}(\mathbb{M}))$ is the set of all $\mathcal{L}(\mathbb{M})$ -sentences satisfied by \mathbb{M} .

Claim. Γ is inconsistent.

Proof of Claim. Assume otherwise. Then, by saturation, Γ is realized by $(\bar{a}, \bar{b}) \in \mathbb{M}$. By homogeneity of \mathbb{M} (implied by saturation), there is an automorphism σ of \mathbb{M} fixing A pointwise and taking \bar{a} to \bar{b} . Since $\mathbb{M} \models \phi(\bar{a}) \wedge \neg\phi(\bar{b})$, σ does not fix X pointwise, contradicting the hypothesis. \square

Hence there are $\mathcal{L}(A)$ -formulas $\psi_1(\bar{x}), \dots, \psi_n(\bar{x})$ such that

$$\mathbb{M} \models \left[\bigwedge_{i=1}^n (\psi_i(\bar{v}) \leftrightarrow \psi_i(\bar{w})) \right] \rightarrow (\phi(\bar{v}) \leftrightarrow \phi(\bar{w})). \quad (*)$$

For $\tau : \{1, 2, \dots, n\} \rightarrow \{0, 1\}$ let $\theta_\tau(\bar{x}) = \bigwedge_{\tau(i)=1} \psi_i(\bar{x}) \wedge \bigwedge_{\tau(i)=0} \neg \psi_i(\bar{x})$. If $\theta_\tau(\bar{a})$ and $\theta_\tau(\bar{b})$ then, by (*), $\bar{a} \in X$ if and only if $\bar{b} \in X$. Thus, if $P = \{\tau : \{1, 2, \dots, n\} \rightarrow \{0, 1\} : \theta_\tau(\bar{a}), \bar{a} \in X\}$, then

$$\bar{a} \in X \text{ if and only if } \mathbb{M} \models \bigvee_{\tau \in P} \theta_\tau(\bar{a}).$$

Hence X is A -definable. □

4 Morley Degree

If X is a definable set of Morley rank α , then X cannot be partitioned into infinitely many sets of Morley rank α . In fact, we show that there exists $d < \omega$ such that X cannot be partitioned into more than d definable sets of Morley rank α .

Proposition 4.1 (Morley degree). *For every \mathbb{M} -definable set X of Morley rank α , there exists $d < \omega$ such that*

- X can be partitioned into d \mathbb{M} -definable subsets of Morley rank α , and
- if Z_1, \dots, Z_n are disjoint \mathbb{M} -definable subsets of X of Morley rank α , then $n \leq d$.

We call d the Morley degree of X and write $\text{MD}(X) = d$. We use the same notation for the degree of the formula defining X .

Proof. We construct a binary tree $S \subseteq 2^{<\omega}$ and $(X_\sigma \subseteq X : \sigma \in S)$ with the following properties.

- (i) If $\sigma \in S$ and $\tau \subseteq \sigma$, then $\sigma \in S$.
- (ii) $X_\emptyset = X$.
- (iii) X_σ is \mathbb{M} -definable and $\text{MR}(X_\sigma) = \alpha$ for all $\sigma \in S$.

- (iv) If $\sigma \in S$, there are two cases. If there is an \mathbb{M} -definable set Y such that $\text{MR}(X_\sigma \cap Y) = \text{MR}(X_\sigma \cap Y^c) = \alpha$, then $\sigma \frown 0, \sigma \frown 1 \in S$ and $X_{\sigma \frown 0} = X_\sigma \cap Y$, $X_{\sigma \frown 1} = X_\sigma \cap Y^c$. If there is no such Y then no $\tau \supset \sigma$ is in S .

We claim that S is finite since otherwise by König's lemma there would be an infinite branch $f : \omega \rightarrow 2$, $f|n \in S$, $\forall n < \omega$, and letting $Y_n = X_{f|n} \cap (X_{f|n+1})^c$, we would get an infinite sequence Y_1, Y_2, \dots of disjoint subsets of X of Morley rank α contradicting $\text{MR}(X) = \alpha$.

Let $S_0 = \{\sigma_0, \dots, \sigma_{d-1}\}$ be the set of terminal nodes of S and $Y_i = X_{\sigma_i}$, $\forall i < d$. Then by construction, $P = \{Y_i : i < d\}$ is a maximal (cannot be refined) partition of X into sets of Morley rank α .

Now if Z_0, \dots, Z_{n-1} are disjoint \mathbb{M} -definable subsets of X of Morley rank α , then by maximality of P for each $i < d$ there exists at most one $j < n$ such that $\text{MR}(Y_i \cap Z_j) = \alpha$. This correspondence defines a partial function from d to n , which is surjective because P covers X . Thus, by Pigeonhole Principle, $n \leq d$. \square

Corollary 4.2. *If $\mathcal{M} \models T$ is \aleph_0 -saturated and X is M -definable of Morley rank α , then X can be partitioned into $\text{MD}(X)$ M -definable subsets of Morley rank α . In particular, there is an M -definable subset of X of Morley rank α and Morley degree one.*

Proof. Let $\phi(\bar{v}, \bar{w})$ be an \mathcal{L} -formula and $\bar{a} \in M$ such that $\phi(\bar{v}, \bar{a})$ defines X . By Proposition 4.1, there are $\psi_1(\bar{v}, \bar{w}_1), \dots, \psi_d(\bar{v}, \bar{w}_d)$ \mathcal{L} -formulas and $\bar{c}_1, \dots, \bar{c}_d \in \mathbb{M}$ such that $\psi_1(\mathbb{M}, \bar{c}_1), \dots, \psi_d(\mathbb{M}, \bar{c}_d)$ form a partition of $\phi(\mathbb{M}, \bar{a})$ and $\text{MR}(\psi_i(\bar{v}, \bar{c}_i)) = \alpha$ for $i \leq d$.

Because \mathcal{M} is \aleph_0 -saturated and $\mathcal{M} \prec \mathbb{M}$, we can find $\bar{b}_1, \dots, \bar{b}_d \in M$ such that $\text{tp}^{\mathcal{M}}(\bar{a}, \bar{b}_1, \dots, \bar{b}_d) = \text{tp}^{\mathbb{M}}(\bar{a}, \bar{c}_1, \dots, \bar{c}_d)$. Thus, by elementarity, $\psi_1(\mathcal{M}, \bar{b}_1), \dots, \psi_d(\mathcal{M}, \bar{b}_d)$ form a partition of $\phi(\mathcal{M}, \bar{a})$ and by Lemma 2.2, $\text{MR}(\psi_i(\bar{v}, \bar{b}_i)) = \alpha$ for all i . \square

Remark. Using Theorem 7.3 (see below), one can show that if T is ω -stable then Corollary 4.2 holds even when \mathcal{M} is not \aleph_0 -saturated.

5 Morley Rank of Types

We naturally extend the definitions of Morley rank and degree to types.

Definition 5.1. If $p \in S_n(A)$ for some $A \subseteq \mathbb{M}$, $|A| < |\mathbb{M}|$, then $\text{MR}(p) = \inf\{\text{MR}(\phi) : \phi \in p\}$.

If $\text{MR}(p)$ is an ordinal (we write $\text{MR}(p) < \infty$), then $\text{MD}(p) = \inf\{\text{MD}(\phi) : \phi \in p \text{ and } \text{MR}(\phi) = \text{MR}(p)\}$.

Thus, for each $p \in S_n(A)$ with $\text{MR}(p) < \infty$, there is an $\mathcal{L}(A)$ -formula ϕ_p (not unique) having the same Morley rank and degree as p .

Lemma 5.2. Let $A \subseteq \mathbb{M}$, $|A| < |\mathbb{M}|$. If $p \neq q \in S_n(A)$ and $\text{MR}(p), \text{MR}(q) < \infty$, then $\phi_p \neq \phi_q$.

Proof. Since $p \neq q$ there exists an $\mathcal{L}(A)$ -formula ψ such that $\psi \in p$ and $\neg\psi \in q$. Hence $\phi_p \wedge \psi \in p$ and $\phi_q \wedge \neg\psi \in q$, and thus by minimality of $\text{MR}(\phi_p)$ and $\text{MR}(\phi_q)$, $\text{MR}(\phi_p \wedge \psi) = \text{MR}(\phi_p) = \text{MR}(p)$ and $\text{MR}(\phi_q \wedge \neg\psi) = \text{MR}(\phi_q) = \text{MR}(q)$. Now if $\phi_p = \phi_q$, $\text{MR}(\phi_p \wedge \psi) = \text{MR}(\phi_q \wedge \neg\psi) = \text{MR}(\phi_p)$, contradicting the minimality of $\text{MD}(\phi_p)$. \square

Corollary 5.3. Let $p \neq q \in S_n(A)$, $A \subset \mathbb{M}$, $|A| < |\mathbb{M}|$, and $\text{MR}(p) = \text{MR}(q) = \alpha$, then $\text{MR}(\phi_p \wedge \phi_q) < \alpha$.

Proof. Assume for contradiction that $\text{MR}(\phi_p \wedge \phi_q) = \alpha$. If $\phi_q \in p$ and $\phi_p \in q$, then we could take ϕ_p and ϕ_q both equal to $\phi_p \wedge \phi_q$ contradicting Lemma 5.2. So, assume WLOG that $\neg\phi_q \in p$. Then $\text{MR}(\phi_p \wedge \neg\phi_q) = \alpha$. By assumption, $\text{MR}(\phi_p \wedge \phi_q) = \alpha$, and hence $\text{MD}(\phi_p \wedge \neg\phi_q) < \text{MD}(\phi_p)$, a contradiction. \square

Now we are ready to prove the following extension of Theorem 1.4.

Theorem 5.4. *TFAE:*

- (i) T is t.t.;
- (ii) T is λ -stable for every $\lambda \geq \aleph_0$;
- (iii) T is ω -stable.
- (iv) T has “finite” Morley rank, i.e. for all $\mathcal{M} \models T$, $\mathcal{L}(M)$ -formula ϕ , $\text{MR}(\phi) < \infty$.

Proof. By Theorem 1.4, it is enough to show the following two implications. **(iv) \Rightarrow (iii):** Assume (iv). Suppose that $|A| \leq \aleph_0$, $A \subseteq M$, $\mathcal{M} \models T$. Since for each $p \in S_n(A)$, $\text{MR}(p) < \infty$, there is ϕ_p as above. Then it follows from Lemma 5.2 that $|S_n(A)| \leq \aleph_0$, and hence T is ω -stable.

(i) \Rightarrow (iv): Assume (iv) doesn't hold, i.e. $\mathcal{M} \models T$ and $\mathcal{L}(M)$ -formula ϕ such that $\text{MR}(\phi) = \infty$. Let $\beta = \sup\{\text{MR}(\psi) : \psi \text{ an } \mathcal{L}(M)\text{-formula and } \text{MR}(\psi) < \infty\}$. Because $\text{MR}(\phi) = \infty \geq \beta + 2$, we can find an $\mathcal{L}(M)$ -formula ψ such that $\text{MR}(\phi \wedge \psi), \text{MR}(\phi \wedge \neg\psi) \geq \beta + 1$ and hence $\text{MR}(\phi \wedge \psi) = \text{MR}(\phi \wedge \neg\psi) = \infty$. Continuing in this manner we construct a binary tree of $\mathcal{L}(M)$ -formulas, and hence T is not t.t. \square

Definition 5.5. For $A \subset \mathbb{M}$ and $\bar{a} \in \mathbb{M}$ define $\text{MR}(\bar{a}/A)$, the Morley rank of \bar{a} over A , to be $\text{MR}(\text{tp}(\bar{a}/A))$. If $A = \emptyset$, we write $\text{MR}(\bar{a})$.

We conclude the section with the following useful lemma.

Lemma 5.6.

- (i) If $X \subseteq \mathbb{M}^n$ is \mathbb{M} -definable, then $\text{MR}(X) = \sup\{\text{MR}(\bar{a}/A) : \bar{a} \in X, A \subset \mathbb{M}, |A| < |\mathbb{M}|, X \text{ is } A\text{-definable}\}$.
- (ii) Let $X \subseteq M^n$ be M -definable with $\text{MR}(X) = \alpha$, where \mathcal{M} is \aleph_0 -saturated. Then for all $\beta < \alpha$, there exists Y_1, Y_2, \dots disjoint M -definable subsets of X of Morley rank β .
- (iii) For any $\mathcal{L}(\mathbb{M})$ -formula ϕ , $|\{p \in S_n(\mathbb{M}) : \phi \in p \text{ and } \text{MR}(p) = \text{MR}(\phi)\}| = \text{MD}(\phi)$.

Proof.

- (i): “ \geq ” is clear. For “ \leq ” let ϕ be an $\mathcal{L}(A)$ -formula defining X , where $A \subset \mathbb{M}$ is finite, and use the satisfiability of $\{\phi\} \cup \{\neg\psi : \psi \text{ an } \mathcal{L}(A)\text{-formula and } \text{MR}(\psi) < \text{MR}(X)\}$ (X cannot be covered by finitely many \mathcal{M} -definable sets of rank smaller than $\text{MR}(X)$) and saturation.
- (ii): Straightforward induction on α using Lemma 2.3.
- (iii): Follows from Proposition 4.1 and Corollary 5.3.

\square

6 Forking

From now on we assume that T is ω -stable (and hence every formula has “finite” Morley rank).

Suppose we have a type $p \in S_n(A)$ and $A \subseteq B \subseteq M$, $\mathcal{M} \models T$. Sometimes it is useful to find $q \in S_n(B)$ with $q \supseteq p$ such that the set of realizations of q is not really smaller than that of p , i.e. q imposes as few as possible restrictions on its realizations. For example, if V is a vector space, $X \subseteq Y \subseteq V$ and $p \in S_1(X)$ is the type of an element independent from X . Then q will be the type of an element independent from Y . Any other type will be more restrictive. This motivates the following definition.

Definition 6.1. *Suppose $A \subseteq B$, $p \in S_n(A)$, $q \in S_n(B)$ and $q \supseteq p$. We say that q is a forking extension of p (or q forks over A) if $\text{MR}(q) < \text{MR}(p)$. If $\text{MR}(q) = \text{MR}(p)$, we say that q is a nonforking extension of p .*

Remark. This is the analogous notion to “wide extensions” introduced in the seminar.

Definition 6.2. *We say $p \in S_n(A)$ is stationary if for all $A \subseteq B$, there is a unique nonforking extension of p in $S_n(B)$.*

Theorem 6.3 (Existence of nonforking extensions). *Suppose that $p \in S_n(A)$ and $A \subseteq B$.*

- (i) *There is $q \in S_n(B)$ a nonforking extension of p .*
- (ii) *There are at most $\text{MD}(p)$ nonforking extensions of p in $S_n(B)$, and if $\mathcal{M} \models T$ is \aleph_0 -saturated with $A \subseteq M$, there are exactly $\text{MD}(p)$ nonforking extensions of p in $S_n(M)$.*
- (iii) *There is at most one $q \in S_n(B)$ nonforking extension of p with $\text{MD}(q) = \text{MD}(p)$. In particular, if $\text{MD}(p) = 1$, then p is stationary.*

Proof. Assume $\text{MR}(p) = \alpha$ and $\text{MD}(p) = d$.

- (i): Let $\psi(\bar{v})$ be an $\mathcal{L}(\mathbb{M})$ -formula defining a subset of $\phi_p(\mathbb{M})$ of Morley rank α and degree one. Set $q = \{\theta(\bar{v}) : \theta \text{ is an } \mathcal{L}(B)\text{-formula and } \text{MR}(\theta \wedge \psi) = \alpha\}$. Because $\text{MR}(\psi) = \alpha$ and $\text{MD}(\psi) = 1$, for any $\mathcal{L}(B)$ -formula χ exactly one of $\chi, \neg\chi$ belongs to q . Thus, since q is clearly finitely satisfiable, $q \in S_n(B)$. Moreover, $p \subseteq q$ and $\text{MR}(q) = \alpha$.

(ii): Suppose $q_1, \dots, q_m \in S_n(B)$ are distinct nonforking extensions of p . Let $\psi_i = \phi_p \wedge \phi_{q_i}$ for $i \leq m$. Then $\text{MR}(\psi_i) = \alpha$ while by Corollary 5.3, $\text{MR}(\psi_i \wedge \psi_j) < \alpha$ for $i, j \leq m$ with $i \neq j$. Thus, by Proposition 4.1, $m \leq \text{MD}(\phi_p) = d$.

If $\mathcal{M} \models T$ is \aleph_0 -saturated with $A \subseteq M$, then by Corollary 4.2, there are $\psi_1(\bar{v}), \dots, \psi_d(\bar{v}) \mathcal{L}(M)$ -formulas such that $\psi_1(\mathcal{M}), \dots, \psi_d(\mathcal{M})$ form a partition of $\phi_p(\mathcal{M})$ and $\text{MR}(\psi_i(\bar{v})) = \alpha$ for $i \leq d$. In particular, $\text{MD}(\psi_i(\bar{v})) = 1$ for $i \leq d$. As in (i), we can find $q_i \in S_n(M)$ extending p and such that $\text{MR}(q_i) = \alpha$ and $\psi_i \in q_i$, for each $i \leq d$. Clearly, q_1, \dots, q_d are distinct.

(iii): Follows from Corollary 5.3. □

7 Definable Types

Recall that T is ω -stable.

Definition 7.1. *We say that a type $p \in S_n(A)$ is definable over B if for each \mathcal{L} -formula $\phi(\bar{v}, \bar{w})$ there exists an $\mathcal{L}(B)$ -formula $d_p(\phi)(\bar{w})$ such that for all $\bar{a} \in A$, $\phi(\bar{v}, \bar{a}) \in p$ if and only if $d_p(\phi)(\bar{a})$.*

The notion of definable types is used to analyze nonforking extensions. Also, it provides a way of generalizing the notion of nonforking extensions to the case of theories that are only λ -stable for some $\lambda \geq \omega$ (not necessarily ω -stable).

To prove the main theorem of this section, we need the following lemma, which is also useful on its own.

Lemma 7.2. *Suppose that \mathcal{M} is \aleph_0 -saturated, $\phi(\bar{v})$ is an $\mathcal{L}(M)$ -formula with $\text{MR}(\phi) = \alpha$, and $\psi(\bar{v})$ is an $\mathcal{L}(\mathbb{M})$ -formula with $\text{MR}(\phi \wedge \psi) = \alpha$. Then there is $\bar{a} \in M$ such that $\mathbb{M} \models \phi \wedge \psi(\bar{a})$.*

Proof. We prove by induction on α . If $\alpha = 0$, then $\phi(\mathbb{M})$ is finite and hence, by elementarity, $\phi(\mathcal{M}) = \phi(\mathbb{M})$. Now assume that $\alpha > 0$. Because of Corollary 4.2, we can assume WLOG that $\text{MD}(\phi) = 1$. Then $\text{MR}(\phi \wedge \neg\psi) = \beta$ for some $\beta < \alpha$. By (ii) of Lemma 5.6, there are $\mathcal{L}(M)$ -formulas $\theta_0(\bar{v}), \theta_1(\bar{v}), \dots$ such that $\text{MR}(\theta_i) = \beta$ and $\theta_0(\mathcal{M}), \theta_1(\mathcal{M}), \dots$ are pairwise disjoint subsets of

$\phi(\mathcal{M})$. Since $\text{MR}(\phi \wedge \neg\psi) = \beta$, $\text{MR}(\neg\psi \wedge \theta_i) < \beta$ for some (in fact for all but finitely many) i . Hence $\text{MR}(\theta_i \wedge \psi) = \beta$ and by induction there is $\bar{a} \in M$ such that $\mathbb{M} \models \theta_i \wedge \psi(\bar{a})$, and thus $\mathbb{M} \models \phi \wedge \psi(\bar{a})$. \square

Theorem 7.3. *Let \mathcal{M} be \aleph_0 -saturated, $\phi(\bar{v})$ be an $\mathcal{L}(A)$ -formula with $A \subseteq M$ and $\text{MR}(\phi) = \alpha$, and $\psi(\bar{v}, \bar{w})$ be an \mathcal{L} -formula. Then the set $B_\psi^\phi = \{\bar{b} \in \mathbb{M} : \text{MR}(\psi(\bar{v}, \bar{b}) \wedge \phi(\bar{v})) = \alpha\}$ is A -definable.*

Proof. By Property 3.2, it is enough to show that B_ψ^ϕ is \mathbb{M} -definable.

Note that without loss of generality we may assume that $d = \text{MD}(\phi) = 1$. Indeed, by Corollary 4.2, there are $\mathcal{L}(M)$ -formulas $\theta_1(\bar{v}), \theta_2(\bar{v}), \dots, \theta_d(\bar{v})$ with Morley rank α such that $\mathcal{M} \models \phi \leftrightarrow \bigvee \theta_i$, and thus $B_\psi^\phi = \bigcup_{i=1}^d B_\psi^{\theta_i}$. Hence B_ψ^ϕ would be definable if each of $B_\psi^{\theta_i}$ is definable.

Now here is the critical claim.

Claim. *For each $\bar{c} \in B_\psi^\phi$ there is a finite set $X_{\bar{c}} \subseteq \phi(\mathcal{M}) \cap \psi(\mathbb{M}, \bar{c})$ such that for all \bar{b} , if $X_{\bar{c}} \subseteq \psi(\mathbb{M}, \bar{b})$, then $\bar{b} \in B_\psi^\phi$.*

Proof of Claim. Assuming otherwise, we inductively construct a sequence $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2, \dots$ of counter examples such that $\bar{a}_i \in \phi(\mathcal{M}) \cap \psi(\mathbb{M}, \bar{c})$ and $\bar{b}_i \notin B_\psi^\phi$, and show that it violates the order property. Assume we have already constructed $\bar{a}_1, \bar{b}_1, \dots, \bar{a}_n, \bar{b}_n$. Then by the inductive hypothesis, $\text{MR}(\phi(\bar{v}) \wedge \bigwedge_{i=1}^n \neg\psi(\bar{v}, \bar{b}_i)) = \alpha$. By Lemma 7.2, there is

$$\bar{a}_{n+1} \in \phi(\mathcal{M}) \cap \psi(\mathbb{M}, \bar{c}) \setminus \bigcup_{i=1}^n \psi(\mathbb{M}, \bar{b}_i).$$

Then, by the assumption, there is $\bar{b}_{n+1} \in \mathbb{M}$ such that $\{\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{n+1}\} \subseteq \psi(\mathcal{M}, \bar{b}_{n+1})$ and $\text{MR}(\phi(\bar{v}) \wedge \psi(\bar{v}, \bar{b}_{n+1})) < \alpha$.

By construction, $\psi(\bar{a}_i, \bar{b}_j)$ if and only if $i \leq j$, therefore, by Lemma 1.6, T has the order property, contradicting Theorem 1.7. \square

Let $Y = \{X \subseteq \phi(\mathcal{M}) : X \text{ is finite and } \forall \bar{b} \in \mathbb{M}, \text{ if } X \subseteq \psi(\mathbb{M}, \bar{b}) \text{ then } \bar{b} \in B_\psi^\phi\}$ and for each $X \in Y$ let $\theta_X(\bar{w}) = \bigvee_{\bar{a} \in X} \psi(\bar{a}, \bar{w})$.

By the claim, $B_\psi^\phi = \bigcup_{X \in Y} \theta_X(\mathbb{M})$, thus B_ψ^ϕ is a “small” ($|Y| \leq |M| < |\mathbb{M}|$) union of definable sets. Because $\text{MD}(\phi(\bar{v})) = 1$, $\mathbb{M} \setminus B_\psi^\phi = B_{\neg\psi}^\phi$. But by the same argument, one can show that $B_{\neg\psi}^\phi$ is a “small” union of definable sets. Thus, by Property 3.1, there is a finite $Y_0 \subseteq Y$ such that $B_\psi^\phi = \bigcup_{X \in Y_0} \theta_X(\mathbb{M})$. Hence B_ψ^ϕ is \mathbb{M} -definable. \square

As a corollary, we get that in ω -stable theories all types are definable.

Corollary 7.4. *If $p \in S_n(A)$ with $A \subset \mathbb{M}$, $|A| < |\mathbb{M}|$, then p is definable over A_0 for some finite $A_0 \subseteq A$.*

Proof. Let $A_0 \subseteq A$ be such that ϕ_p is an $\mathcal{L}(A_0)$ -formula and A_0 is finite, and let $\alpha = \text{MR}(\phi_p)$. Then, for any \mathcal{L} -formula $\psi(\bar{v}, \bar{w})$ and $\bar{a} \in A$, $\psi(\bar{v}, \bar{a}) \in p$ if and only if $\text{MR}(\phi_p(\bar{v}) \wedge \psi(\bar{v}, \bar{a})) = \alpha$. Now apply Theorem 7.3. \square

8 Independence

Forking can be used to give a notion of independence in ω -stable theories.

Definition 8.1. *We say that \bar{a} is independent from B over A if $\text{tp}(\bar{a}/A \cup B)$ does not fork over A . We write $\bar{a} \downarrow_A B$.*

With this notion of independence we get all of the intuitive properties we would want to have.

Lemma 8.2.

- (i) *Monotonicity:* If $\bar{a} \downarrow_A B$ and $C \subseteq B$, then $\bar{a} \downarrow_A C$.
- (ii) *Transitivity:* $\bar{a} \downarrow_A \bar{b}$ and $\bar{a} \downarrow_{A\bar{b}} \bar{c}$ if and only if $\bar{a} \downarrow_A \bar{b}\bar{c}$.
- (iii) *Finite Basis:* $\bar{a} \downarrow_A B$ if and only if $\bar{a} \downarrow_A B_0$ for all finite $B_0 \subseteq B$.
- (iv) *Symmetry:* If $\bar{a} \downarrow_A \bar{b}$, then $\bar{b} \downarrow_A \bar{a}$.

Proof. We prove only Symmetry since the rest of the statements are immediate from the definitions.

Let $\alpha = \text{MR}(\bar{a}/A)$ and $\beta = \text{MR}(\bar{b}/A)$, and suppose that $\text{MR}(\bar{a}/A\bar{b}) = \text{MR}(\bar{a}/A) = \alpha$. We need to show that $\text{MR}(\bar{b}/A\bar{a}) = \text{MR}(\bar{b}/A) = \beta$.

We first assume that $A = M$, where M is the universe of an \aleph_0 -saturated \mathcal{M} . Let $\phi(\bar{v}) \in \text{tp}(\bar{a}/M)$ with $\text{MR}(\phi) = \alpha$ and let $\psi(\bar{w}) \in \text{tp}(\bar{b}/M)$ with $\text{MR}(\psi) = \beta$. Assume for contradiction that there is an $\mathcal{L}(M)$ -formula $\theta(\bar{v}, \bar{w})$ such that $\mathbb{M} \models \theta(\bar{a}, \bar{b})$ and $\text{MR}(\theta(\bar{a}, \bar{w})) < \beta$. By Theorem 7.3, there is an $\mathcal{L}(M)$ -formula $\chi(\bar{v})$ defining $\{\bar{x} \in \mathbb{M} : \text{MR}(\psi(\bar{w}) \wedge \theta(\bar{x}, \bar{w})) < \beta\}$. Clearly, $(\phi(\bar{v}) \wedge \chi(\bar{v})) \wedge \theta(\bar{v}, \bar{b}) \in \text{tp}(\bar{a}/M\bar{b})$, and since $\text{MR}(\bar{a}/M\bar{b}) = \alpha$, $\text{MR}((\phi(\bar{v}) \wedge \chi(\bar{v})) \wedge \theta(\bar{v}, \bar{b})) = \alpha$. Thus, by Lemma 7.2, there is $\bar{a}' \in M$ such that $\mathbb{M} \models (\phi(\bar{a}') \wedge \chi(\bar{a}')) \wedge \theta(\bar{a}', \bar{b})$ and hence $\theta(\bar{a}', \bar{w}) \in \text{tp}(\bar{b}/M)$ and $\text{MR}(\psi(\bar{w}) \wedge \theta(\bar{a}', \bar{w})) < \beta$, contradicting $\text{MR}(\bar{b}/M) = \beta$.

For the general case, let \mathcal{M} be an \aleph_0 -saturated model containing A . Let $\bar{b}' \in \mathbb{M}$ realize a nonforking extension of $\text{tp}(\bar{b}/A)$ to M , and hence $\text{MR}(\bar{b}'/M) = \beta$. Because \mathbb{M} is saturated, there is $\bar{a}' \in \mathbb{M}$ such that $\text{tp}(\bar{a}\bar{b}/A) = \text{tp}(\bar{a}'\bar{b}'/A)$. Thus it follows from Lemma 2.2 that $\text{MR}(\bar{a}'/A\bar{b}') = \text{MR}(\bar{a}/A\bar{b}) = \alpha$.

Let $\bar{a}'' \in \mathbb{M}$ realize a nonforking extension of $\text{tp}(\bar{a}'/A\bar{b}')$ to $M\bar{b}'$, and thus $\text{MR}(\bar{a}''/M\bar{b}') = \text{MR}(\bar{a}'/A\bar{b}') = \alpha$. On the other hand, $\text{tp}(\bar{a}/A) = \text{tp}(\bar{a}'/A) = \text{tp}(\bar{a}''/A)$ and hence $\alpha = \text{MR}(\bar{a}/A) = \text{MR}(\bar{a}''/A) \geq \text{MR}(\bar{a}''/M) \geq \text{MR}(\bar{a}''/M\bar{b}') = \alpha$. Thus $\text{MR}(\bar{a}''/M) = \text{MR}(\bar{a}''/M\bar{b}')$, so by the first part of the proof, $\text{MR}(\bar{b}'/M\bar{a}'') = \text{MR}(\bar{b}'/M) = \beta$.

Since $\text{tp}(\bar{a}\bar{b}/A) = \text{tp}(\bar{a}'\bar{b}'/A) = \text{tp}(\bar{a}''\bar{b}'/A)$, it follows from Lemma 2.2 that $\text{MR}(\bar{b}/A\bar{a}) = \text{MR}(\bar{b}'/A\bar{a}'') \geq \text{MR}(\bar{b}'/M\bar{a}'') = \beta$. Thus $\text{MR}(\bar{b}/A\bar{a}) = \text{MR}(\bar{b}/A)$. \square

9 Morley Sequences

Recall that T is ω -stable.

Definition 9.1. *Suppose that $p \in S_1(A)$ and δ is an ordinal. We say that $(a_\alpha : \alpha < \delta)$ is a Morley sequence for p over A if for each $\alpha < \delta$ the type $\text{tp}(a_\alpha/A \cup \{a_\beta : \beta < \alpha\})$ is an extension of p of the same Morley rank and degree.*

In particular, each a_α is a realization of p with $a_\alpha \perp_A \{a_\beta : \beta < \alpha\}$. Recall from (iii) of Theorem 6.3 that for any $A \subseteq B$ there is at most one nonforking extension of p in $S_1(B)$ of the same Morley degree, and if $\text{MD}(p) = 1$, p is stationary.

We first show that Morley sequences are sets of indiscernibles.

Theorem 9.2. *If $I = (a_\alpha : \alpha < \delta)$ is an infinite Morley sequence for p over A , then I is a set of indiscernibles over A .*

Proof. By Theorem 1.9, it suffices to show that I is a sequence of order indiscernibles. Let $d = \text{MD}(p)$. We will show by induction on n that $\text{tp}(a_{\alpha_1}, \dots, a_{\alpha_n}/A) = \text{tp}(a_{\beta_1}, \dots, a_{\beta_n}/A)$ if $\alpha_1 < \dots < \alpha_n < \delta$ and $\beta_1 < \dots < \beta_n < \delta$.

For $n = 1$, a_{α_1} and a_{β_1} are both realizations of p , and hence $\text{tp}(a_{\alpha_1}/A) = p = \text{tp}(a_{\beta_1}/A)$.

Assume that $\text{tp}(a_{\alpha_1}, \dots, a_{\alpha_{n-1}}/A) = \text{tp}(a_{\beta_1}, \dots, a_{\beta_{n-1}}/A)$. Then by homogeneity of \mathbb{M} , there is an automorphism σ fixing A and such that $\sigma(\alpha_i) = \beta_i$ for $i < n$. Since $a_{\alpha_n} \downarrow_A \{a_\beta : \beta < \alpha_n\}$, $a_{\alpha_n} \downarrow_A \{a_{\alpha_1}, \dots, a_{\alpha_{n-1}}\}$ and hence a_{α_n} realizes the unique nonforking extension of p to $A \cup \{a_{\alpha_1}, \dots, a_{\alpha_{n-1}}\}$ of Morley degree d . Thus, because of Lemma 2.2, $\sigma(a_{\alpha_n})$ realizes the unique nonforking extension of p to $A \cup \{a_{\beta_1}, \dots, a_{\beta_{n-1}}\}$ of Morley degree d . But a_{β_n} also realizes the latter unique extension, and hence

$$\text{tp}(a_{\alpha_1}, \dots, a_{\alpha_n}/A) = \text{tp}(\sigma(a_{\alpha_1}), \dots, \sigma(a_{\alpha_n})/A) = \text{tp}(a_{\beta_1}, \dots, a_{\beta_n}/A).$$

□

As an application of Morley sequences we show (without using any combinatorial results such as Ramsey's theorem) that all uncountable models of ω -stable theories contain infinite sets of indiscernibles.

Theorem 9.3. *If $\mathcal{M} \models T$, $A \subset M$, $|A| < |M|$, and $|M| \geq \aleph_1$, then there is an infinite set $I \subset M$ of indiscernibles over A .*

Proof. We can assume WLOG that $\kappa = |M|$ is a successor cardinal (and hence regular) because if κ was a limit cardinal, using downward Lowenheim-Skolem we could instead work in $\mathcal{N} \prec \mathcal{M}$ with $A \subseteq N$, $|N| < \kappa$ successor.

Our goal is to construct a Morley sequence over A and use Theorem 9.2.

Because T is ω -stable, if $B \subset M$ and $|B| < \kappa$ then $|S_1(B)| < \kappa$, and since κ is regular, there is $p \in S_1(B)$ such that p has κ many realizations. Call such a p large. Let $(\gamma, d) = \inf\{(\text{MR}(p), \text{MD}(p)) : p \in S_1(B), p \text{ is large}, A \subseteq B \subset M, |B| < \kappa\}$, where \inf is taken in the lexicographic order. Let $A_0 \supseteq A$ and $p \in S_1(A_0)$ be such that $\text{MR}(p) = \gamma$ and $\text{MD}(p) = d$.

Claim.

- (i) If $A_0 \subseteq B \subset M$ and $|B| < \kappa$, then there is a unique $p_B \in S_1(B)$ with $p \subseteq p_B$ large.
- (ii) $\text{MR}(p_B) = \gamma$ and $\text{MD}(p_B) = d$.
- (iii) If $A_0 \subseteq B \subseteq C \subset M$ and $|C| < \kappa$, then $p_C \supseteq p_B$.

Proof of Claim. Because $|B| < \kappa$, $S_1(B) < \kappa$, and since p is large, there is $p_B \in S_1(B)$ with $p_B \supseteq p$ such that p_B is large. Clearly, $\text{MR}(p_B) \leq \text{MR}(p) = \gamma$. On the other hand, by the choice of γ , $\text{MR}(p_B) \geq \gamma$, and hence $\text{MR}(p_B) = \gamma$. Therefore, by the choice of d , $\text{MR}(p_B) = d$. Thus, by (iii) of Theorem 6.3, p_B is the unique nonforking extension of p of degree d .

If $A_0 \subseteq B \subseteq C \subset M$ and $|C| < \kappa$, then $p_C|B$ is a large extension of p . Thus, by (i), $p_C|B = p_B$. \square

Using the claim, we construct a Morley sequence for p over A_0 as follows. Given $(a_\alpha : \alpha < \delta)$, let $A_\delta = A_0 \cup \{a_\alpha : \alpha < \delta\}$ and let a_δ realize p_{A_δ} . Such a_δ exists since p_{A_δ} is large. By the claim, $(p_{A_\alpha} : \alpha < \kappa)$ is an increasing sequence of extensions of p of Morley rank γ and degree d . Thus $(a_\alpha : \alpha < \kappa)$ is a Morley sequence for p over A_0 , and hence is a set of indiscernibles over A_0 (and therefore over A), by Theorem 9.2. \square