1 An Infinitary Setting

We will begin with an infinite sequence of groups G_p and subsets $X_p \subseteq G_p$ such that for each p, $|X_p X_p^{-1} X_p| \leq K|X_p|$ for some fixed K independent of p.

Take an extension of the language of groups which adds predicates $m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$ whenever ϕ is a formula and r a rational, and a distinguished predicate X. Then for each p, (G_p, X_p) gives a model of this language, interpreting $m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$ to hold when $\frac{|\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}|}{|X^{(|\vec{x}|)}|} \geq r$.

Take an ultraproduct of these models, giving a model (G, X) with G, X infinite; in particular, G is countably saturated. We may define a measure μ on G-definable subsets of $G^{(l)}$: if $A = \{\vec{x} \mid \phi(\vec{x}, \vec{a})\}$, set $\mu(A) := \inf_r (G, X_0) \vDash m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$.

If $C \subseteq G^{(n+1)}$ is a definable set, we define $C_{\vec{a}} := \{x \mid (x, \vec{a}) \in C\}$.

Note that if $A = \{\vec{x} \mid \phi(\vec{x}, \vec{a})\}$, it is not always the case that $\mu(A) \geq r$ is equivalent to $(G, X_0) \vDash m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$: if $(G, X_0) \vDash m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq s$ for every rational $s > r$ then we will have $\mu(A) = r$ even if $(G, X) \vDash \neg m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$.

In particular, $\{a \mid \mu(C_a) > 0\}$ may not be definable. However, $\mu(C_a)$ is $\bigwedge$-definable: $\mu(C_a) > 0$ iff $(G, X) \vDash \neg m(C_a) \geq r$ for all $r > 0$.

Rather than work with G, we are more interested in $\tilde{G} := \bigcup_n (X^{-1}X)^n$. From here on, we will restrict to the model $(\tilde{G}, X)$. Fix a countable elementary submodel $M \prec \tilde{G}$; for the remainder, we will always allow parameters from M.

The definable sets give a countable, finitely additive algebra of subsets of $\tilde{G}$, and μ is a finitely additive measure on them. The definable sets on $\tilde{G}^{(2)}$ are also a countable, finitely additive algebra, but much bigger than the product of the collection of definable subsets of $\tilde{G}$ with itself. Since we have sufficient saturation, we could extend μ to a σ-additive measure on the σ-algebra generated by the definable sets, and these measures have the following convenient properties:

- $\mu(A \times B) = \mu(A)\mu(B)$
- If $\mu(A_a) = 0$ for all a then $\mu(A) = 0$

We have $\mu(X) = 1$, and $\mu(X^{(n)})$ is bounded for each n.

Definition 1. *A* partial type *p over a set D is a collection of $M \cup D$-definable subsets of $\tilde{G}$ such that whenever $\mathcal{A} \subseteq p$ is finite, $\bigcap \mathcal{A}$ is non-empty. A* type *over D is a maximal partial type over D. A* global type *is a type over $\tilde{G}$.*

A partial type p is wide *if whenever $\mathcal{A} \subseteq p$ is finite, $\mu(\bigcap \mathcal{A}) > 0$.*

We write $a \vDash p$ if for every $A \in p$, $a \in A$. We write $p(\tilde{G}) := \bigcap p = \{a \mid a \vDash p\}$. We write $tp(a/D)$ for the collection of $M \cup D$-definable sets containing a.

In particular, $tp(a/D)$ is the unique type such that $a \vDash tp(a/D)$. We will only allow the case where D (and therefore $M \cup D$) is countable.

Note that being a type is relative to a particular choice of algebra: in general, a wide typeover a is only a wide partial type over a, b.

Lemma 2. *Each type is contained in $X^{(n)}$ for a fixed n, or disjoint from all $X^{(n)}$.*

Proof. This follows from the fact that $X^{(n)}$ is definable for each n. □

From here on, we always assume that a type is contained in $X^{(n)}$ for some n. By countable saturation, all such types are realized.

Lemma 3. *Every wide partial type can be extended to a wide type.*

Proof. Let $B_1, \ldots, B_i, \ldots$ enumerate the definable sets. Define a sequence of partial types with $q_0 := q$ and, given q_n, if for some $A \supseteq q_n$, $\mu(A \cap B_n) = 0$, $q_{n+1} := q_n \cup \{\overline{B}_n\}$, otherwise $q_{n+1} := q_n \cup \{B_n\}$.

Let $q' := \bigcup_n q_n$. If q' is not wide, there must be some n such that q_{n+1} is not wide. Let n be least such that q_{n+1} is not wide. If $B_n \in q_{n+1}$ then there is some $A \in q_n$ with $\mu(B_n \cap A) = 0$; this contradicts the definition of q_{n+1}. So $\overline{B}_n \in q_{n+1}$, and there is an $A \in q_n$ with $\mu(A \cap B_n) = 0$. But there must be some $A' \in q_n$ with $\mu(A' \cap \overline{B}_n) = 0$. Then $\mu(A \cap A') = 0$, contradicting the fact that $A, A' \in q_n$ and q_n is wide. Therefore q' is wide. □

If $A, B \subseteq \tilde{G}$ (either definable or $\bigwedge$-definable), we write $Ab = \{ab \mid a \in A\}$, $bA = \{ba \mid a \in A\}$, $AB = \{ab \mid a \in A, b \in B\}$, $A^{-1} = \{a^{-1} \mid a \in A\}$. Similarly, we write $q^{-1} := \{A^{-1} \mid A \in q\}$, $qA := \{BA \mid B \in q\}$, $Aq := \{AB \mid B \in q\}$, $pq := \{AB \mid A \in p, B \in q\}$.

Lemma 4.

- *If q is a type, so is q^{-1}*
- *If q is a type over D and $a \in D$, $qa = \{xa \mid x \in q\}$ is a type over D.*

Proof.

- For any definable B, either $B^{-1} \in q$ or $\overline{B^{-1}} \in q$, and therefore eitehr $B \in q^{-1}$ or $\overline{B} \in q^{-1}$ respectively.
- For any B, either $Ba^{-1} \in q$ or $\overline{B}a^{-1} \in q$, and therefore either $B \in qa$ or $\overline{B} \in qa$ respectively.

□

Lemma 5. *If q is a type,*

- $q^{-1}(\tilde{G}) = (q(\tilde{G}))^{-1}$
- $(qA)(\tilde{G}) = (q(\tilde{G}))A$
- $(pq)(\tilde{G}) = (p(\tilde{G}))(q(\tilde{G}))$

Proof. Only the third requires proof. If $x \in (p(\tilde{G}))(q(\tilde{G}))$ then $x \in AB$ for all $A \in p$, $B \in q$, and therefore $x \in (pq)(\tilde{G})$. Conversely, if $x \in (pq)(\tilde{G})$, so $x \in AB$ whenever $A \in p, B \in q$, it follows by saturation that there is a pair $(y, z) \in p(\tilde{G}) \times q(\tilde{G})$ with $yz = x$, and therefore $x \in p(\tilde{G})q(\tilde{G})$. □

Note that if we allowed types over uncountable sets, we would need more than countable saturation to make the previous lemma work.

2 Wide Types

Definition 6. (b_i) *is a* sequence of indiscernibles *if whenever* $\phi(b_0, \ldots, b_n)$ *and* $m_0 < \cdots < m_n$, $\phi(b_{m_0}, \ldots, b_{m_n})$.

That is, for every n there is a type p_n of $n+1$-tuples such that $tp(b_{m_0}, \ldots, b_{m_n}) = p_n$ whenever $m_0 < \cdots < m_n$ is an increasing sequence.

Lemma 7. *Suppose $\mu(A_b) > 0$ holds and (b_i) is a sequence of indiscernibles with $tp(b_i) = tp(b)$. Then for any finite $b_1, \ldots, b_n$, there is an $a \in \bigcap_{i \leq n} A_{b_i}$.*

Proof. Suppose not: that is, for some k, $\bigcap_{i \leq k} A_{b_i}$ is empty. Let k be least such that $\mu(\bigcap_{i \leq k+1} A_{b_i}) = 0$; since $\mu(A_{b_0}) > 0$, $k \geq 0$. For $n \geq k$, let $C_n = \bigcap_{i<k} A_{b_i} \cap A_{b_n}$. Then $\mu(C_n) = \mu(\bigcap_{i \leq k} A_{b_i}) > 0$ since $k < k+1$. But for $n < m$, by indiscernibility, $\mu(C_n \cap C_m) = \mu(\bigcap_{i \leq k+1} A_{b_i}) = 0$. This is a contradiction since μ is a finite measure. □

Lemma 8. *If $tp(a/b)$ is wide and (b_i) is a sequence of indiscernibles with $tp(b_i) = tp(b)$ then for any finite $b_1, \ldots, b_n$, there is an a' with $tp(a', b_i) = tp(a, b)$ for all $i \leq n$.*

Proof. Suppose $(a, b) \in A$. Then $a \in A_b$, so by the previous lemma, there is an a' so that $a' \in \bigcap_{i \leq n} A_{b_i}$, so $(a', b_i) \in A$ for each $i \leq n$. By saturation, there is an a' such that $(a', b_i) \in A$ simultaneously for all $i \leq n$, $A \ni (a, b)$. Therefore $tp(a', b_i) = tp(a, b)$ for all $i \leq n$. □

Definition 9. *A global type p is M-invariant if whenever $tp(a_1, \ldots, a_n) = tp(a'_1, \ldots, a'_n)$ and $A_{a_1,\ldots,a_n} \in p$, also $A_{a'_1,\ldots,a'_n} \in p$. p is M-finitely satisfiable if whenever $\mathcal{A} \subseteq p$ is finite, $\bigcap \mathcal{A} \cap M \neq \emptyset$.*

Lemma 10. *If p is finitely satisfiable in M then p is M-invariant.*

Proof. Suppose not; then there are $a_1, \ldots, a_n, a'_1, \ldots, a'_n$ so that $A_{a_1,\ldots,a_n} \setminus A_{a'_1,\ldots,a'_n} \in p$. But then there is an element $m \in M$ such that $m \in A_{a_1,\ldots,a_n} \setminus A_{a'_1,\ldots,a'_n}$, contradicting the fact that $tp(a_1, \ldots, a_n) = tp(a'_1, \ldots, a'_n)$. □

Lemma 11. *If p is a type over M, there is an extension of p to a global M-finitely satisfiable (and therefore M-invariant) type.*

Proof. Let $p' := p \cup \{\overline{A} \mid A \cap M = \emptyset\}$; if p' is not finitely satisfiable in $\tilde{G}$, there is an $A \in p$ and a B with $B \cap M = \emptyset$ such that $A \cap \overline{B} = \emptyset$. But since $A \in p$ and p is a type over M, $A \neq \emptyset$, and since M is an elementary submodel of $\tilde{G}$, $A \cap M \neq \emptyset$. So $A \cap M \cap (B \cup \overline{B}) \neq \emptyset$, and since $B \cap M = \emptyset$, $A \cap \overline{B} \neq \emptyset$.

Let q be an arbitrary extension of p' to a type. Suppose q is not finitely satisfiable in M; then there is a $B \in q$ with $B \cap M = \emptyset$, which is impossible since then $\overline{B} \in p' \subseteq q$. □

Lemma 12. *If p is a wide type over M, there is an extension of p to a wide global M-finitely satisfiable (and therefore M-invariant) type.*

Proof. Let $p' := p \cup \{\overline{A} \mid \mu(A \cap M) = 0\}$; if p' is not finitely satisfiable in $\tilde{G}$, there is an $A \in p$ and a B with $\mu(B \cap M) = 0$ such that $\mu(A \cap \overline{B}) = 0$. But since $A \in p$ and p is a wide type over M, $\mu(A) > 0$, and since M is an elementary submodel of $\tilde{G}$, $\mu(A \cap M) > 0$. So $\mu(A \cap M \cap (B \cup \overline{B})) > 0$, and since $\mu(B \cap M) = 0$, $\mu(A \cap \overline{B}) > 0$.

Let q be an arbitrary extension of p' to a type. Suppose q is not finitely satisfiable in M; then there is a $B \in q$ with $B \cap M = \emptyset$, which is impossible since then $\overline{B} \in p' \subseteq q$. Similarly, if q is not wide, there is a $B \in q$ with $\mu(B) = 0$, and therefore $\mu(B \cap M) = 0$, again a contradiction. □

Definition 13. *If S is a set and p a global type, write $p \mid S$ for the restriction of p to sets definable over S.*

Lemma 14. *Suppose p is a global M-invariant type, and recursively choose $b_n \vDash p \mid M \cup \{b_i\}_{i<n}$. Then $\{b_i\}$ is a sequence of indiscernibles.*

Proof. By induction on n, we show $tp(b_{m_0}, \ldots, b_{m_n})$ is constant whenever $m_0 < \cdots < m_n$. When $n = 1$, this follows since each $b_i \in p(\tilde{G})$. Suppose $(b_{n+1}, b_n, \ldots, b_0) \in A$. Then also $(b_{m_{n+1}}, b_n, \ldots, b_0) \in A$ holds because $tp(b_{m_{n+1}}/b_0, \ldots, b_n) = tp(b_{n+1}/b_0, \ldots, b_n)$. By IH, $tp(b_0, \ldots, b_n) = tp(b_{m_0}, \ldots, b_{m_n})$. Since $A_{b_n,\ldots,b_0} \in p$, also $A_{b_{m_n},\ldots,b_{m_0}} \in p$, so $(b_{m_{n+1}}, b_{m_n}, \ldots, b_{m_0}) \in A$. □

Lemma 15. *Suppose q is an M-invariant type, $b \vDash q \mid M \cup \{a\}$, $tp(b') = tp(b)$, $tp(a') = tp(a)$, $tp(a'/b)$ is wide, and $\mu(A_a \cap B_b) > 0$. Then $\mu(A_{a'} \cap B_b) > 0$.*

Proof. Suppose the claim fails, so $\mu(A_{a'} \cap B_{b'}) = 0$. By countable saturation and the fact that $tp(b') = tp(b)$, there an a'' with $tp(a'', b) = tp(a', b')$, so we may assume $b = b'$. There is some $\epsilon > 0$ so that whenever $\hat{a} \vDash tp(a)$ and $\hat{b} \vDash q \mid M \cup \{\hat{a}\}$, $\mu(A_{\hat{a}} \cap B_{\hat{b}}) \geq \epsilon$. Choose $a_0 \vDash p$ and $b_0 \vDash q \mid a_0$. Given $\{a_i, b_i\}_{i \leq n}$, since $tp(a'/b)$ is wide, $\mu(\{x \mid \mu(A_x \cap B_b) = 0\})$ is $\bigwedge$-definable, and therefore there is an a_{n+1} with $tp(a_{n+1}) = tp(a)$ such that $\mu_x(A_{a_{n+1}} \cap B_{b_i}) = 0$ for $i \leq n$, and let $b_{n+1} \vDash q \restriction \{a_i, b_i\}_{i \leq n}, \{a_{n+1}\}$. For each i, let $C_i = A_{a_i} \cap B_{b_i}$. Since each $b_n \vDash q \restriction a_n$, $\mu(C_i) \geq \epsilon$. This is a contradiction. □

Lemma 16. *Suppose q is an M-invariant type, $b \vDash q \mid M \cup \{a\}$, $tp(b') = tp(b)$, $tp(a') = tp(a)$, $tp(a'/b)$ is wide, and $\mu(A_a \cap B_b) = 0$. Then $\mu(A_{a'} \cap B_b) = 0$.*

Proof. Suppose the claim fails, so $\mu(A_{a'} \cap B_{b'}) > 0$. By countable saturation and the fact that $tp(b') = tp(b)$, there an a'' with $tp(a'', b) = tp(a', b')$, so we may assume $b = b'$. Given $\{a_i, b_i\}_{i<n}$ (including the empty case where $n = 0$), choose $b_n \vDash q \mid \{a_i, b_i\}_{i<n}$. Since $\mu(\{x \mid \mu(A_x \cap B_b) \geq \delta\})$ is $\bigwedge$-definable, there is an $a_n \vDash p$ such that $\mu(A_{a_n} \cap B_{b_n}) \geq \delta$. Then, setting $C_i := A_{a_i} \cap B_{b_i}$, $\mu(C_i) \geq \delta$, and when $i < j$, $\mu(C_i \cap C_j) = 0$ since $b_j \vDash q' \mid a_i$, so $\mu(A_{a_i} \cap B_{b_j}) = 0$. □

Theorem 17. *Let p, q be types (over M), and let $a, a' \in p(\tilde{G})$, $b, b' \in q(\tilde{G})$ with $tp(a/b), tp(a'/b')$ wide. Then $\mu(A_a \cap B_b) > 0$ iff $\mu(A_{a'} \cap B_{b'}) > 0$.*

Proof. Suppose $\mu(A_a \cap B_b) \geq \delta > 0$ but $\mu(A_{a'} \cap B_{b'}) = 0$. Fix an extension q' of q to an invariant global type; let $\hat{a} \vDash p$ and $\hat{b} \vDash q' \mid M \cup \{\hat{a}\}$. If $\mu(A_{\hat{a}} \cap B_{\hat{b}}) > 0$ then by Lemma 15, both $\mu(A_a \cap B_b) > 0$ and $\mu(A_{a'} \cap B_{b'}) > 0$. Otherwise, $\mu(A_{\hat{a}} \cap B_{\hat{b}}) = 0$, and by Lemma 16, both $\mu(A_a \cap B_b) = 0$ and $\mu(A_{a'} \cap B_{b'}) = 0$. □

3 The "Stabilizer" Theorem

Lemma 18. *If q is a type, p is a wide type, and $a, b \in q(\tilde{G})$, $tp(a/b)$ wide, then $pa \cap pb$ is wide.*

Proof. Suppose $pa \cap pb$ is not wide; then there is a definable $A \in p$ with $\mu(A) > 0$ such that $\mu(Aa \cap Ab) = 0$. Let $a_0, \ldots, a_n, \ldots$ be a sequence of elements in $q(\tilde{G})$ with $tp(a_n/a_0, \ldots, a_{n-1})$ wide for each n. Then by Theorem 17, $\mu(Aa_i \cap Aa_j) = 0$ for all $i \neq j$, which is impossible since $\mu(Aa_n) = \mu(A)$ for all n. □

Lemma 19. *Let q be a type, let p be a wide type, let $Q := \{a^{-1}b \mid a, b \in q(\tilde{G}), tp(b/a) \text{ wide}\}$. Then whenever $c_1 \in q^{-1}q(\tilde{G})$, $c_2 \in Q$, and $tp(c_2/c_1)$ wide, $pc_1^{-1} \cap pc_2^{-1}$ is wide.*

Proof. Let $r_i := tp(c_i)$. By Theorem 17, it suffices to show the claim for some $c_1 \models r_1, c_2 \models r_2$ with $tp(c_2/c_1)$ wide. Let $a_0 \models q$; then there is some $a_1 \models q$ such that $a_0^{-1}a_1 \models r_1$. Since $c_2 \in Q$, there is an $a_2' \models q$ such that $tp(a_2'/a_0)$ is wide and $a_0^{-1}a_2' \models r_2$. We may extend $tp(a_2'/a_0)$ to a wide type over a_0, a_1, and let a_2 realize this type. Since $tp(a_2/a_0, a_1)$ is wide, so is $tp(a_0^{-1}a_2/a_0, a_1)$, and therefore so is $tp(a_0^{-1}a_2/a_0^{-1}a_1)$. We have $a_0^{-1}a_i \models r_i$.

By Lemma 18, $pa_1^{-1} \cap pa_2^{-1}$ is wide, and by invariance, $pa_1^{-1}a_0 \cap pa_2^{-1}a_0$ is wide as well. □

Lemma 20. *Let q be a type, let p be a wide type, let $Q := \{a^{-1}b \mid a, b \in q(\tilde{G}), tp(b/a) \text{ wide}\}$. Then whenever $c_1 \in q^{-1}q(\tilde{G})$, $c_2 \in Q$, and $tp(c_1/c_2)$ wide, $pc_2 \cap pc_1^{-1}$ is wide.*

Proof. We have $c_2^{-1} \models q^{-1}q$ as well, and since $tp(c_2/c_1)$ is wide, in particular $tp(c_2)$ is wide, so $tp(c_2^{-1})$ is wide. Since M is an elementary submodel and $tp(c_2^{-1})$ wide, $tp(c_2^{-1})$ extends to a wide global invariant type r and let $c \models r \mid c_1$.

Since $c_1 = a^{-1}b$ where $tp(b/a)$ is wide, it follows that $tp(a^{-1}b/a)$ is wide, so also $tp(c_1) = tp(a^{-1}b)$ is wide. Let $\hat{c}_1 \models tp(c_1)$ with $tp(\hat{c}_1/c)$ wide. By Lemma 19, $p\hat{c}_1^{-1} \cap pc^{-1}$ is wide. By Lemma 16, it follows that $pc_1^{-1} \cap pc^{-1}$ is wide.

Since $tp(c/c_1)$ and $tp(c_2^{-1}/c_1)$ are both wide, by Theorem 17, $pc_1^{-1} \cap pc_2$ is wide. □

Lemma 21. *Let q be a wide type, let $Q := \{a^{-1}b \mid a, b \in q(\tilde{G}), tp(b/a) \text{ wide}\}$. Let $a \models q^{-1}q$, $b_1, \ldots, b_n \in Q$, and $tp(a/b_1, \ldots, b_n)$ wide. Then $qa \cap q(b_1 \cdots b_n)^{-1}$ is wide.*

Proof. By induction on n. Since $tp(a/b_1)$ is wide, $qa \cap qb_1^{-1}$ is wide, so by Lemma 20 with $p = q$, there are $c_0, c_1 \models q$ such that $c_0a = c_1b_1^{-1}$, and therefore $ab_1 = c_0c_1$. So $ab_1 \models q^{-1}q$, and since $tp(a/b_1, \ldots, b_n)$ is wide, so is $tp(ab_1/b_1, \ldots, b_n)$, and therefore $tp(ab_1/b_2, \ldots, b_n)$; so by induction, $qab_1 \cap q(b_2 \cdots b_n)^{-1}$ is wide. Multiplying by b_1^{-1} gives $qa \cap q(b_1 \cdots b_n)^{-1}$ is wide. □

Lemma 22. *If q is a wide type and $Q := \{a^{-1}b \mid a, b \in q(\tilde{G}), tp(b/a) \text{ wide}\}$, $Q^n \subseteq q^{-1}qq^{-1}q(\tilde{G})$.*

Proof. Let $b_1, \ldots, b_n \in Q$. Choose any $a \models q^{-1}q$ with $tp(a/b_1, \ldots, b_n)$ wide. Then by Lemma 22, $qa \cap q(b_1 \cdots b_n)^{-1}$ is wide, and choosing any $c, d \models q$ such that $ca = d(b_1 \cdots b_n)^{-1}$, we have $ab_1 \cdots b_n = c^{-1}d \models q^{-1}q$. Then $b_1 \cdots b_n = a^{-1}(ab_1 \cdots b_n) \models q^{-1}qq^{-1}q$. □

Definition 23. *We say a wide type q has* Property F *if for every $a \models q$, there is a $b \models q$ with $tp(a/b)$ and $tp(b/a)$ both wide.*

If B is a set of pairs, write $B^\top := \{(y, x) \mid (x, y) \in B\}$.

Lemma 24. *Let C, D be definable sets of pairs, and let P be a definable set with $\mu(P) > 0$. Assume $P \times P \subseteq C \cup D$. Then there is a definable $P' \subseteq P$ with $\mu(P') > 0$ and either:*

- $\mu(C_a) > 0$ *for every $a \in P'$, or*
- $\mu(D_a^\top) > 0$ *for every $a \in P'$*

Proof. Let $Q_1 \subseteq Q_2 \subseteq \cdots$ be a sequence of subsets of P such that $a \in Q_n$ implies $\mu(C_a) > 0$[1]. If for some n, $\mu(Q_n) > 0$, we may take $P' := Q_n$. If $\mu(Q_n) = 0$ for all n, it follows by the Fubini property of μ that $\mu(C \cap (\tilde{G} \times P)) = 0$.

Similarly, either there is a $P' \subseteq P$ $\mu(P') > 0$ and $\mu(D_a^\top) > 0$ for every $a \in P'$, or $\mu(D \cap (P \times \tilde{G})) = 0$.

If neither of these possibilities for P' exist, we have

$$\mu(P)^2 = \mu(P \times P) = \mu((C \cup D) \cap (P \times P)) \leq \mu(C \cap (\tilde{G} \times P)) + \mu(D \cap (P \times \tilde{G})) = 0$$

contradicting the assumption. □

Theorem 25. *If B is a set with $\mu(B) > 0$, there is a wide type q with $B \in q$ such that q has property F.*

Proof. By choosing an ordering of the definable sets, and an ordering of the pairs of definable sets in which each pair appears infinitely often, we may choose a wide type $q \ni B$ such that whenever $P \in q$, $P \times P \subseteq C \cup D$, either for every $a \in P$ $\mu(C_a) > 0$, or for every $a \in P$ $\mu(D_a^\top) > 0$.

Let $a \in q$, and let $q_0 := q \cup \{\overline{C_a} \mid \mu(C_a) = 0\} \cup \{\overline{D_a} \mid \mu(D_a^\top) = 0\}$. First, we must check that q_0 is consistent: if not, we have $P \cap \overline{C_a} \cap \overline{D_a} = \emptyset$ for some $P \in q$ and some C, D with $\mu(C_a) = \mu(D_a^\top) = 0$. Then $P \subseteq C_a \cup D_a$; since q is a type, $P' := \{x \mid P \subseteq C_x \cup D_x\} \in q$, and therefore setting $P'' := P \cap P'$, we have $P'' \in q$ and whenever $x, y \in P''$, $y \in C_x \cup D_x$. Equivalently, $P'' \times P'' \subseteq C \cup D$. By the construction of q, there is a $Q \in q$ such that either for every $x \in Q$, $\mu(C_x) > 0$, or for every $x \in Q$, $\mu(D_x^\top) > 0$. In particular, $a \in Q$, contradicting the fact that $\mu(C_a) = \mu(D_a^\top) = 0$.

Since q_0 is consistent, it may be extended to a type q' over a. Let $b \in q'$; if $C(b, a)$ holds, $b \in C_a$, and if $\mu(C_a) = 0$, we would have $\overline{C_a} \in q_0 \subseteq q'$. So $\mu(C_a) > 0$. Therefore $tp(b/a)$ is wide.

On the other hand, suppose $D(b, a)$ holds, so $b \in D_a$, and therefore $\mu(D_a^\top) > 0$. Since $tp(b) = tp(a)$, it follows that $\mu(D_b^\top) > 0$ as well, so $tp(a/b)$ is wide also. □

[1] Such a sequence exists by the $\bigwedge$-definability of μ; for instance, take $Q_n := \{a \in P \mid m(C_a) > 1/n\}$.

Lemma 26. *If q is a wide type with property F and $Q := \{a^{-1}b \mid a, b \in q(\tilde{G}), tp(b/a) \text{ wide}\}$, whenever $c, d \models q$, $d^{-1}c \in QQ$.*

Proof. Given $c, d \models q$, let $b \models q$ be such that $tp(b/c)$ and $tp(c/b)$ are wide. Extend $tp(b/c)$ to a type r wide over c, d, and let a realize this type. Then $tp(a/c, d)$ and $tp(c/a)$ are both wide, so $a^{-1}c, d^{-1}a \in Q$, so $d^{-1}c \in QQ$. □

Theorem 27. *If q is a wide type with property F then $q^{-1}qq^{-1}q(\tilde{G})$ is a group.*

Proof. By Lemma 26, $(q^{-1}qq^{-1}q(\tilde{G}))^2 \subseteq Q^8$ where $Q := \{a^{-1}b \mid a, b \in q(\tilde{G}), tp(b/a) \text{ wide}\}$ and by Lemma 22, $Q^8 \subseteq q^{-1}qq^{-1}q(\tilde{G})$. □

We will write $[q, q]$ for the type $q^{-1}qq^{-1}q$ and $S := [q, q](\tilde{G})$.

Lemma 28. *If T and U are wide subgroups then $T \cap U$ is wide.*

Proof. Let $V := T \cap U$, and suppose V is not wide. Choose a maximal set $u_1, \ldots, u_n, \ldots$ from U such that $Vu_i \cap Vu_j = \emptyset$ for $i \neq j$. Since V is not wide, this set is infinite. But then $Tu_i \cap Tu_j = \emptyset$ for $i \neq j$, which is impossible since T is wide. □

Lemma 29. *If T is a wide group and r is a type, $r(\tilde{G})$ is contained in a single coset of T.*

Proof. Suppose not; let $a, b \models r$ so that $aT \cap bT = \emptyset$; there must be a definable $A \supseteq T$ with $\mu(A) > 0$ but $aA \cap bA = \emptyset$. Let r' be a global invariant type extending r, and consider the sequence given by $c_n \models r' \mid \{a, b\} \cup \{c_i\}_{i<n}$. Since $a, b \models r \mid \emptyset$, the sequences $a, c_0, \ldots$ and $b, c_0, \ldots$ are both indiscernible. In particular, either $aA \cap c_0A = \emptyset$ or $bA \cap c_0A = \emptyset$; in either case, $c_0A, \ldots, c_nA, \ldots$ give infinitely many disjoint copies of A, which is impossible since $\mu(A) > 0$. So $r(\tilde{G})$ is contained in a single coset of S. □

Lemma 30. *If q is a wide type with property F and $T \subseteq [q, q](\tilde{G})$ is a wide $\bigwedge$-definable subgroup then $T = S$.*

Proof. Set $S := [q, q](\tilde{G})$. Since $q^{-1}q(\tilde{G}) \subseteq S$, $q(\tilde{G}) \subseteq aS$ for any $a \models q$. Let $r := q[q, q]$ and $R := r(\tilde{G})$. Since T is a subgroup of S, T acts on R by right multiplication. By the same argument as the previous lemma, q is contained in a single coset of T. That is, $q(\tilde{G}) \subseteq aT$ for some $a \models q$. It follows that $q^{-1}(\tilde{G})q(\tilde{G}) \subseteq T$, and therefore $S \subseteq T$. □

Lemma 31. *If q is a wide type with property F then S is a normal subgroup of $\tilde{G}$.*

Proof. Let r be a type in $\tilde{G}$. By Lemma 29, $r(\tilde{G})$ is contained in a single coset of S; call it C_r. In particular, the conjugate group $S' := C_r^{-1}SC_r$ is $\bigwedge$-definable and, since it is conjugate to S, wide. So $S \cap S'$ is wide, so by the previous lemma, $S \cap S' = S$. So for any $a \models r$, $S \subseteq a^{-1}Sa$, which implies that $aSa^{-1} \subseteq S$. Since this holds for every $a \models r$, and every type r, S is normal. □

4 An Application

Write $a^X := \{x^{-1}ax \mid x \in X\}$, $A^{(l)}$ for l-tuples from A.

Theorem 32 (Corollary 1.2)**.** *For any k, l, m, there is a $p < 1$ and a K such that whenever G is a group, $X_0 \subseteq G^{p,k}$ finite, and $X = X_0^{-1}X_0$:*

- $|X_0X| \leq k|X_0|$
- *With probability $\geq p$, an l-tuple $(a_1, \ldots, a_l) \in (X)^{(l)}$ satisfies $|a_1^X \cdots a_l^X| \geq |X|/m$*

Then there is a subgroup S of G, $S \subseteq X^2$, such that X is contained in $\leq K$ cosets of S.

Suppose not; then for some k, l, m and every p, K, there is a $G^{p,k}$, an $X_0^{p,k} \subseteq G^{p,k}$ finite, such that the conditions hold, but there is no definable subgroup S with X contained in K cosets of S.

So $\{(a_1, \ldots, a_l) \in X^{(l)} \mid \mu(a_1^X \cdots a_l^X) \geq 1/m\}$ is not definable. However, if we choose $r \in (1/(m+1), 1/m)$, and set $Q = \{(a_1, \ldots, a_l) \in X^{(l)} \mid m(a_1^X \cdots a_l^X) \geq r\}$, Q is definable, and we will have $\mu(Q) = 1$ and $\mu(a_1^X \cdots a_l^X) \geq 1/m+1$ whenever $(a_l, \ldots, a_l) \in Q$.

We may refine X_0 to a type q to which the work above applies, so $S := q^{-1}qq^{-1}q$ is a normal subgroup of $\tilde{G}$.

Pick a countable sequence of equivalence relations $E_1, \ldots, E_n, \ldots$ so that $tp(a) = tp(b)$ iff $a \equiv_{E_n} b$ for all n, each E_i has finitely many equivalence classes, and E_j refines E_i for $i < j$ (for instance, each E_n has the form $a \equiv_{E_n} b$ iff $\phi_i(a) \leftrightarrow \phi_i(b)$ for $i \leq n$, where the ϕ_i are an enumeration of formulas).

In particular, for each i, there is an equivalence class F_i with $\mu(X_0 \cap F_i) > 0$. Let $C_i := F_i^{-1}F_i \cap X$; so $\mu(C_i) > 0$ as well, and therefore $\mu(C_i^{(l)}) > 0$. For each i, $\mu(C_i^{(l)} \cap Q) > 0$, and by saturation, there is an $(a_1, \ldots, a_l) \in Q$ where each $a_j \in C_i = F_i^{-1}F_i \cap X$. This means $a_j = b_j^{-1}c_j$ where $b_j \equiv_{E_i} c_j$ for each i. Therefore $tp(b_j) = tp(c_j)$, and, by Lemma 29, $b_jS = c_jS$. Therefore $a_j \in S$ for each j. Since S is normal, $a_1^X \cdots a_l^X \subseteq S$, and therefore S has fewer than $\mu(X_0X)(m+1)$ distinct cosets.

So X_0/S is finite and since $\mu(XX) \leq k|\mu(X_0)|$, XX/S is finite as well. This means that XX is the union of finitely many translations of S: $XX = S \cup \bigcup_i c_iS$.

But this means that both S and the complement of S are $\bigwedge$-definable. If there were no j with $S_j = S$ then we would have $S_j \cap \overline{S}$ non-empty for all j, and therefore $S \cap \overline{S}$ non-empty by saturation; this is clearly impossible, so $S = S_j$ for some j. Therefore S is definable and finitely many cosets of S cover X. This is a contradiction, so there must have been some p, K satisfying the theorem.