1 An Infinitary Setting

We will begin with an infinite sequence of groups G_p and subsets $X_p \subseteq X$ such that for each p, $|X_pX_pX_p| \leq K|X_p|$ for some fixed K independent of p. (In actual examples, we may modify the exact properties satisfied by X_p.)

Take an extension of the language of groups which adds predicates $m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$ whenever ϕ is a formula and r a rational, and a distinguished predicate X. Then for each p, (G_p, X_p) gives a model of this language, interpreting $m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$ to hold when $\frac{|\{\vec{x}|\phi(\vec{x},\vec{a})\}|}{|X^{(|\vec{x}|)}|} \geq r$.

Take an ultraproduct of these models, giving a model (G, X) with G, X infinite. We may define a measure μ on G-definable subsets of $G^{(l)}$: if $A = \{\vec{x} \mid \phi(\vec{x}, \vec{a})\}$, set $\mu(A) := \inf_r (G, X_0) \vDash m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$.

If $C \subseteq G^2$ is a definable set, we define $C_a := \{x \mid (x, a) \in C\}$.

Note that if $A = \{\vec{x} \mid \phi(\vec{x}, \vec{a})\}$, it is not always the case that $\mu(A) \geq r$ is equivalent to $(G, X_0) \vDash m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$: if $(G, X_0) \vDash m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq s$ for every rational $s > r$ then we will have $\mu(A) = r$ even if $(G, X) \vDash \neg m(\{\vec{x} \mid \phi(\vec{x}, \vec{a})\}) \geq r$.

In particular, $\{a \mid \mu(C_a) > 0\}$ may not be definable. However, $\mu(C_a)$ is $\bigwedge$-definable: $\mu(C_a) > 0$ iff $(G, X) \vDash \neg m(C_a) \geq r$ for all $r > 0$.[1]

Rather than work with G, we are more interested in $\tilde{G} := \bigcup_n (X^{-1}X)^n$. From here on, we will restrict to the model $(\tilde{G}, X)$. Fix a countable elementary submodel $M \prec \tilde{G}$; for the remainder, we will always allow parameters from M.

The M-definable sets give a countable, finitely additive algebra of subsets of $\tilde{G}$, and μ is a finitely additive measure on them. The M-definable sets on $\tilde{G}^{(2)}$ are also a countable, finitely additive algebra, but much bigger than the product of the collection of definable subsets of $\tilde{G}$ with itself. Since we have sufficient saturation, we could extend μ to a σ-additive measure on the σ-algebra generated by the definable sets, and the measure over $\tilde{G}^{(2)}$ would satisfy Fubini's Theorem relative to the measure over $\tilde{G}$; however we will be able to work with the following more elementary properties:

- $\mu(A \times B) = \mu(A)\mu(B)$
- If $\mu(A_a) = 0$ for all a then $\mu(A) = 0$

We have $\mu(X) = 1$, and $\mu(X^{(n)})$ is bounded.

Definition 1. *A* partial type *p over a set D is a collection of $M \cup D$-definable subsets of $\tilde{G}$ such that whenever $\mathcal{A} \subseteq p$ is finite, $\bigcap \mathcal{A}$ is non-empty. A* type *over D is a maximal partial type over D. A* global type *is a type over $\tilde{G}$.*

A type p is wide *if whenever $\mathcal{A} \subseteq p$ is finite, $\mu(\bigcap \mathcal{A}) > 0$.*

We write $a \vDash p$ if for every $A \in p$, $a \in A$. We write $p(\tilde{G}) := \bigcap p = \{a \mid a \vDash p\}$. We write $tp(a/D)$ for the collection of $M \cup D$-definable sets containing a.

In particular, $tp(a/D)$ is the unique type such that $a \vDash tp(a/D)$.

Note that being a type is relative to a particular choice of algebra: in general, a wide typeover a is only a wide partial type over a, b.

[1] Here we abuse notation by writing $m(C_a)$, which is not quite a formula of the language.

Lemma 2. *Each type is contained in $X^{(n)}$ for a fixed n, or disjoint from all $X^{(n)}$.*

Proof. This follows from the fact that $X^{(n)}$ is definable for each n. □

From here on, we always assume that a type is contained in $X^{(n)}$ for some n. By saturation, all such types are realized.

Lemma 3. *Every wide partial type can be extended to a wide type.*

Proof. Let $B_1, \ldots, B_i, \ldots$ enumerate the definable sets. Define a sequence of partial types with $q_0 := q$ and, given q_n, if for some $A \supseteq q_n$, $\mu(A \cap B_n) = 0$, $q_{n+1} := q_n \cup \{\overline{B}_n\}$, otherwise $q_{n+1} := q_n \cup \{B_n\}$.

Let $q' := \bigcup_n q_n$. If q' is not wide, there must be some n such that q_{n+1} is not wide. Let n be least such that q_{n+1} is not wide. If $B_n \in q_{n+1}$ then there is some $A \in q_n$ with $\mu(B_n \cap A) = 0$; this contradicts the definition of q_{n+1}. So $\overline{B}_n \in q_{n+1}$, and there is an $A \in q_n$ with $\mu(A \cap B_n) = 0$. But there must be some $A' \in q_n$ with $\mu(A' \cap \overline{B}_n) = 0$. Then $\mu(A \cap A') = 0$, contradicting the fact that $A, A' \in q_n$ and q_n is wide. Therefore q' is wide. □

If $A, B \subseteq \tilde{G}$ (either definable or $\bigwedge$-definable), we write $Ab = \{ab \mid a \in A\}$, $bA = \{ba \mid a \in A\}$, $AB = \{ab \mid a \in A, b \in B\}$, $A^{-1} = \{a^{-1} \mid a \in A\}$. Similarly, we write $q^{-1} := \{A^{-1} \mid A \in q\}$, $qA := \{BA \mid B \in q\}$, $Aq := \{AB \mid B \in q\}$, $pq := \{AB \mid A \in p, B \in q\}$.

Lemma 4.
- *If q is a type, so is q^{-1}*
- *If q is a type over D and $a \in D$, $qa = \{xa \mid x \in q\}$ is a type over D.*

Proof.
- For any definable B, either $B^{-1} \in q$ or $\overline{B^{-1}} \in q$, and therefore eitehr $B \in q^{-1}$ or $\overline{B} \in q^{-1}$ respectively.
- For any B, either $Ba^{-1} \in q$ or $\overline{B}a^{-1} \in q$, and therefore either $B \in qa$ or $\overline{B} \in qa$ respectively.

□

Lemma 5.
- $q^{-1}(\tilde{G}) = (q(\tilde{G}))^{-1}$
- $(qA)(\tilde{G}) = (q(\tilde{G}))A$
- $(pq)(\tilde{G}) = (p(\tilde{G}))(q(\tilde{G}))$

Proof. Only the third requires proof. If $x \in (p(\tilde{G}))(q(\tilde{G}))$ then $x \in AB$ for all $A \in p$, $B \in q$, and therefore $x \in (pq)(\tilde{G})$. Conversely, if $x \in (pq)(\tilde{G})$, so $x \in AB$ whenever $A \in p, B \in q$, it follows by saturation that there is a pair $(y, z) \in p(\tilde{G}) \times q(\tilde{G})$ with $yz = x$, and therefore $x \in p(\tilde{G})q(\tilde{G})$. □

2 Wide Types are Special

Definition 6. (b_i) *is a* sequence of indiscernibles *if whenever* $\phi(b_0, \dots, b_n)$ *and* $m_0 < \dots < m_n$, $\phi(b_{m_0}, \dots, b_{m_n})$.

That is, for every n there is a type p_n of $n+1$-tuples such that $tp(b_{m_0}, \dots, b_{m_n}) = p_n$ whenever $m_0 < \dots < m_n$ is an increasing sequence.

Lemma 7. *Suppose* $\mu(A_b) > 0$ *holds and* (b_i) *is a sequence of indiscernibles with* $tp(b_i) = tp(b)$. *Then for any finite* $b_1, \dots, b_n$, *there is an* $a \in \bigcap_{i \leq n} \phi_{b_i}$.

Proof. Suppose not: that is, for some k, $\bigcap_{i \leq k} A_{b_i}$ is empty. Let k be least such that $\mu(\bigcap_{i \leq k+1} A_{b_i}) = 0$; since $\mu(A_{b_0}) > 0$, $k \geq 0$. For $n \geq k$, let $C_n = \bigcap_{i<k} A_{b_i} \cap A_{b_n}$. Then $\mu(C_n) = \mu(\bigcap_{i \leq k} A_{b_i}) > 0$ since $k < k+1$. But for $n < m$, by indiscernibility, $\mu(C_n \cap C_m) = \mu(\bigcap_{i \leq k+1} A_{b_i}) = 0$. This is a contradiction since μ is a finite measure. □

Lemma 8. *If* $tp(a/b)$ *is wide and* (b_i) *is a sequence of indiscernibles with* $tp(b_i) = tp(b)$ *then for any finite* $b_1, \dots, b_n$, *there is an* a' *with* $tp(a', b_i) = tp(a, b)$ *for all* $i \leq n$.

Proof. Suppose $(a, b) \in A$. Then $a \in A_b$, so by the previous lemma, there is an a' so that $a' \in \bigcap_{i \leq n} A_{b_i}$, so $(a', b_i) \in A$ for each $i \leq n$. By saturation, there is an a' such that $(a', b_i) \in A$ simultaneously for all $i \leq n$, $A \ni (a, b)$. Therefore $tp(a', b_i) = tp(a, b)$ for all $i \leq n$. □

Definition 9. *A type* p *is* M*-invariant if whenever* σ *is an automorphism preserving* M *and* $A_a \in p$ *then* $A_{\sigma(a)} \in p$. p *is* M*-finitely satisfiable if whenever* $\mathcal{A} \subseteq p$ *is finite,* $\bigcap \mathcal{A} \cap M \neq \emptyset$.

Lemma 10. *If* p *is finitely satisfiable in* M *then* p *is* M*-invariant.*

Proof. Suppose not; then there is an automorphism σ and an a so that $A_a \setminus A_{\sigma(a)} \in p$. But then there is an element $m \in M$ such that $m \in A_a \setminus A_{\sigma(a)}$, contradicting the fact that σ is an automorphism preserving M. □

Lemma 11. *If* p *is a type over* M, *there is an extension of* p *to a global* M*-finitely satisfiable (and therefore* M*-invariant) type.*

Proof. Let $p' := p \cup \{\overline{A} \mid A \cap M = \emptyset\}$; if p' is not finitely satisfiable in $\tilde{G}$, there is an $A \in p$ and a B with $B \cap M = \emptyset$ such that $A \cap \overline{B} = \emptyset$. But since $A \in p$ and p is a type over M, $A \neq \emptyset$, and since M is an elementary submodel of $\tilde{G}$, $A \cap M \neq \emptyset$. So $A \cap M \cap (B \cap \overline{B}) \neq \emptyset$, and since $B \cap M = \emptyset$, $A \cap \overline{B} \neq \emptyset$.

Let q be an arbitrary extension of p' to a type. Suppose q is not finitely satisfiable in M; then there is a $B \in q$ with $B \cap M = \emptyset$, which is impossible since then $\overline{B} \in p' \subseteq q$. □

Definition 12. *If* S *is a set and* p *a global type, write* $p \mid S$ *for the restriction of* p *to sets definable over* S.

Lemma 13. *Suppose p is a global M-invariant type, and recursively choose $b_n \vDash p \mid M \cup \{b_i\}_{i<n}$. Then $\{b_i\}$ is a sequence of indiscernibles.*

Proof. By induction on n, we show $tp(b_{m_0}, \ldots, b_{m_n})$ is constant whenever $m_0 < \cdots < m_n$. When $n = 1$, this follows since each $b_i \in p(\tilde{G})$. Suppose $\phi(b_0, \ldots, b_n, b_{n+1})$. Then also $\phi(b_0, \ldots, b_n, b_{m_{n+1}})$ holds because $tp(b_{m_{n+1}}/b_0, \ldots, b_n) = tp(b_{n+1}/b_0, \ldots, b_n)$. By IH, $tp(b_0, \ldots, b_n) = tp(b_{m_0}, \ldots, b_{m_n})$, and therefore there is an automorphism σ fixing M such that $\sigma(b_i) = b_{m_i}$ for all $i \leq n$. Since $\phi(b_0, \ldots, b_n, x) \in p$, also $\phi(b_{m_0}, \ldots, b_{m_n}, x) \in p$, so $\phi(b_{m_0}, \ldots, b_{m_n}, b_{m_{n+1}})$. □

Theorem 14. *Let p, q be types (over M), and let $a, a' \in p(\tilde{G})$, $b, b' \in q(\tilde{G})$ with $tp(a/b), tp(a'/b')$ wide. Then $\mu(A_a \cap B_b) > 0$ iff $\mu(A_{a'} \cap B_{b'}) > 0$.*

Proof. Suppose $\mu(A_a \cap B_b) > 0$ but $\mu(A_{a'} \cap B_{b'}) = 0$. Consider an automorphism σ with $\sigma(b') = b$; then $tp(\sigma(a')/b)$ is wide and $\mu(A_{\sigma(a')} \cap B_b) = 0$. So we may assume $b = b'$.

Fix an extension q' of q to a global type. Let $a_0 := a$, $b_0 := b$. Given $\{a_i, b_i\}_{i \leq n}$, since $tp(a'/b)$ is wide, $\mu(\{x \mid \mu(A_x \cap B_b) = 0\})$ is $\bigwedge$-definable, and therefore there is an a_{n+1} with $tp(a_{n+1}) = tp(a)$ such that $\mu_x(A_{a_{n+1}} \cap B_{b_i}) = 0$ for $i \leq n$, and let $b_{n+1} \vDash q \restriction \{a_i, b_i\}_{i \leq n}, \{a_{n+1}\}$.

For each i, let $C_i = A_{a_i} \cap B_{b_i}$. Clearly $\mu(C_i \cap C_j) = 0$ when $i \neq j$. Since for any $i < j$, there is an automorphism σ mapping a_i to a_j and fixing M, and therefore preserving q, it follows that $\mu(C_i) = \mu(C_0) > 0$ for all i. This is a contradiction. □

Lemma 15. *If q is a wide type, p is a wide type, and $a, b \in q$, $tp(a/b)$ wide, then $pa \cap pb$ is wide.*

Proof. Suppose $pa \cap pb$ is not wide; then there is a definable $A \in p$ with $\mu(A) > 0$ such that $\mu(Aa \cap Ab) = 0$. Let $a_0, \ldots, a_n$ be a sequence of elements from q with $tp(a_n/a_0, \ldots, a_{n-1})$ wide for each n. Then $\mu(Aa_i \cap Aa_j) = 0$ for all $i \neq j$; but this is impossible since $\mu(Aa_i)$ is constantly non-zero. □