Mathematical Research and the Internet
Mahler lecture series
Terence Tao (UCLA)
Mathematics is an ancient subject…

Ishango bone (10,000 BCE or earlier)

Euclid’s Elements (300 BCE)
Academic publishing (17th century)

...and the way we do maths has not changed much over the years...

Blackboards (circa 1800)
...until recently.
New technologies are changing the way we do maths research.

Many of these technologies are internet-based.
Many have no practical equivalent in the past.

“Google in the 1960s”
The internet has greatly increased a mathematician’s ability to

1. learn existing mathematics;
2. communicate new developments;
   and
3. collaborate to produce new mathematics.

(and also to illustrate presentations.)
Mathematician A wants to understand a paper in field B, but it is full of B’s technical jargon, and no expert on B is available.

It used to be quite difficult to get the basic information needed to read the paper.
It’s significantly easier to do so now…
(though there is still room for improvement)
...even if the information is in an obscure location or a foreign language.
It is not just papers, books, and definitions one can find online… there is also a lot of “folklore” and other informal discussion.

“The school of Athens”, Raphael
Thanks to the internet, the type of insights once reserved for seminars or conference hallways can now be preserved and accumulated.

“The school of Athens”, Raphael
Determinantal processes

23 August, 2009 in expository, math.PR, math.QA, math.SP | Tags: GUE, point processes, random matrices | by Terence Tao | 12 comments (Edit)

Given a set $S$, a (simple) point process is a random subset $A$ of $S$. (A non-simple point process would allow multiplicity; more formally, $A$ is no longer a subset of $S$, but is a Radon measure on $S$, where we give $S$ the structure of a locally compact Polish space, but I do not wish to dwell on these sorts of technical issues here.) Typically, $A$ will be finite or countable, even when $S$ is uncountable. Basic examples of point processes include

- (Bernoulli point process) $S$ is an at most countable set, $0 \leq p \leq 1$ is a parameter, and $A$ a random set such that the events $x \in A$ for each $x \in S$ are jointly independent and occur with a probability of $p$ each. This process is automatically simple.
Blogging is a flexible medium, which can support anything from mathematical lecture notes...
Why global regularity for Navier-Stokes is hard

18 March, 2007 in expository, math.AP, opinion, question | Tags: critical equations, Navier-Stokes equations, scale invariance, turbulence | by Terence Tao | 131 comments

It is always dangerous to venture an opinion as to why a problem is hard (cf. Clarke’s first law), but I’m going to stick my neck out on this one, because (a) it seems that there has been a lot of effort expended on this problem recently, sometimes perhaps without full awareness of the main difficulties, and (b) I would love to be proved wrong on this opinion :-) .

The global regularity problem for Navier-Stokes is of course a Clay Millennium Prize problem and it would be redundant to describe it again here. I will note, however, that it asks for existence of global smooth solutions to a Cauchy problem for a nonlinear PDE. There are countless other global regularity results of this type for many (but certainly not all) other nonlinear PDE; for instance, global regularity is known for Navier-Stokes in two spatial dimensions rather than three (this result essentially dates all the way back to Leray’s thesis in 1933!). Why is the three-dimensional Navier-Stokes global regularity problem considered so hard, when global regularity for so many other equations is easy, or at least achievable?

(For this post, I am only considering the global regularity problem for Navier-Stokes, from a purely mathematical viewpoint, and in the precise formulation given by the Clay Institute; I will not discuss at all the question as to what implications a rigorous solution (either positive or negative) to this problem would have for physics, computational fluid dynamics, or other disciplines, as these are beyond my area of expertise. But if anyone qualified in these fields wants to make a comment along these lines, by all means do so.)

Read the rest of this entry »

... to expository articles ...
The blue-eyed islanders puzzle

Given that there has recently been a lot of discussion on this blog about this logic puzzle, I thought I would make a dedicated post for it (and move all the previous comments to this post). The text here is adapted from an earlier page of mine from a few years back.

The puzzle has a number of formulations, but I will use this one:

There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

[Added, Feb 15: for the purposes of this logic puzzle, "highly logical" means that any conclusion that can logically be deduced from the information and observations available to an islander, will automatically be known to that islander.]

Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 999 of the 1000 tribespeople).

One day, a blue-eyed foreigner visits the island and wins the complete trust of the tribe.
Blogs are interactive; in many cases, the comments are at least as valuable as the main article.
There are dozens of other research maths blogs out there.
Maths wikis (such as the DispersiveWiki, above) can provide detailed information on a specialised subject.
On the largest and most well known wiki – Wikipedia – the quality of the maths articles is steadily increasing.
The Tricki (launched this year) aims to store the tricks that mathematicians need in their daily work, but don’t teach explicitly in textbooks.
There are also a handful of high-quality mathematical video presentations online.
Communicating new developments

An amazing mathematical breakthrough has just been made. How can you find out more about it?
Prior to the internet, one had to
1. Talk to the right person;
2. Be in the right place;
3. Get hold of a physical preprint; or
4. Wait for the result to be published.
Nowadays, one could subscribe to the arXiv…
In view of the speculation on the status of my work on the Taniyama-Shimura conjecture and Fermat's Last Theorem I will give a brief account of the situation. During the review process a number of problems emerged, most of which have been resolved, but one in particular I have not yet solved. The key reduction of (most cases of) the Taniyama-Shimura conjecture to the calculation of the Selmer group is correct. However the final calculation of a precise upper bound for the Selmer group in the semistable case (of the symmetric square representation associated to a modular form) is not yet complete as it stands. I believe that I will be able to finish this in the near future using the ideas explained in my Cambridge lectures.

The fact that a lot of work remains to be done on the manuscript makes it still unsuitable for release as a preprint. In my course in Princeton beginning in February I will give a full account of this work.

Andrew Wiles.
Hassell’s proof of scarring for the Bunimovich stadium

Last year, as part of my "open problem of the week" series (now long since on hiatus), I featured one of my favorite problems, namely that of establishing scarring for the Bunimovich stadium. I’m now happy to say that this problem has been solved (for generic stadiums, at least, and for phase space scarring rather than physical space scarring) by my old friend (and fellow Aussie), Andrew Hassell, in a recent preprint. Congrats Andrew!

Actually, the argument is beautifully simple and short (the paper is a mere 9 pages), though it of course uses the basic theory of eigenfunctions on domains, such as Weyl’s law, and I can give the gist of it here (suppressing all technical details).

Let’s first recall the problem. We consider a stadium domain formed by adjoining two semicircles on the ends of a rectangle, like so:

![Stadium diagram]

We can normalise the rectangle to have height 1 and width t, and will call this stadium $S_t$. For reasons that will be clearer later, it is convenient to view $t$ as a time parameter, so that the stadium is steadily getting elongated in time. The Laplacian on this domain (with Dirichlet boundary conditions) has a countable sequence of eigenfunctions $u_1, u_2, \ldots$ associated to an increasing sequence of eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, which we can normalise so that $\int_{S_t} |u_k|^2 = 1$ for all $k$. The conjecture is that the $u_k$ do not equidistribute in $S_t$.
Collaboration and the internet

Maths collaborations used to be inconveniently slow. And rare.
This is changing.

The internet is one of the reasons for this.

Email (and LaTeX) have obviously made long-distance collaboration substantially easier and faster...

\begin{theorem}
\label{mainthm}
The prime numbers contain infinitely many arithmetic progressions of length $k$ for all $k$.
\end{theorem}
as have several other, less well known, technologies.
But the internet is not only useful for organising existing collaborations…

“I finally got my hands on the Gromov paper and read it carefully. I put some notes about the argument on the wiki, as well as a scanned PDF…”

“I’ve updated the open problems page using the log from our chat. I think we can now make headway on Problem 4…”

“Oops! You’re right, the ergodic theory approach doesn’t work after all. I’ve reverted the online draft back to the June 6 version, but of course we can undo this if we find a way to fix the problem…”

“I searched our blog archives and found that we had actually considered the induction strategy two years ago! But now that we have the counting lemma, I think we can make our old arguments work at last…”
“Your blog post is fascinating! The issue you describe seems analogous to the following phenomenon in algebraic geometry...”

“Hi, I saw your interesting preprint on the arXiv yesterday. I had a thought about Remark 2.3...”

“... it is helping to create new ones.

“... Unfortunately, I don’t know enough about high-dimensional geometry to tell if this approach is feasible using current technology. Perhaps the readers here have some intuition on the problem?”

“You may want to check out Gowers’ latest post on his blog. He has a way of looking at the regularity lemma which could help with the question in your comment...”
For instance, this year the first massively collaborative mathematical projects (or “polymath” projects) were launched.
The first project immediately attracted hundreds of comments.
Several blog posts and a wiki dedicated to the problem were quickly set up.
Dozens of volunteers spontaneously collaborated on the project.
After six weeks and almost a thousand comments, the problem was solved.
Writing up the results is taking a little longer, though.
Further polymath projects are currently underway.
The future

There are many promising technologies out there to help do research better.

“Future City” by Jonathan Stephens
In some ways, there are *too* many such technologies. And they don’t always work well with each other.

“We are stuck with technology when what we really want is just stuff that works.” – Douglas Adams
But these issues should fade with time as later generations of tools become easier to use, more integrated, and more mainstream.
Eventually, some version of these tools will be as universally adopted among mathematicians as email and LaTeX are today.
Technology Adoption

Source: Forbes Magazine
Further reading
(on the internet, of course)

- [http://golem.ph.utexas.edu/category/2009/08/what_do_mathematicians_need_to.html](http://golem.ph.utexas.edu/category/2009/08/what_do_mathematicians_need_to.html) - a discussion on what mathematicians need to know about blogging
- [http://michaelnielsen.org/blog/doing-science-online/](http://michaelnielsen.org/blog/doing-science-online/) - an essay on how new technologies are changing science