Mathematics 245A
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Final, March 13, 2009

Instructions: Do all seven problems; they are all of equal value. The exam should be returned to me by noon of Monday, March 16.

This is a take-home exam. References other than the class textbook, lecture notes, or your own notes can be used so long as they are cited. Communication with other students regarding the exam before the due date is not allowed. Please feel free to email me any questions concerning the final, though I of course cannot provide hints.

You can either use the space provided for the final, or else staple your own work to the final before handing it in.

Good luck!

Name: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points). __________

Problem 2 (10 points). __________

Problem 3 (10 points). __________

Problem 4 (10 points). __________

Problem 5 (10 points). __________

Problem 6 (10 points). __________

Problem 7 (10 points). __________

Total (70 points): ____________________

Problem 1. For any $1 \leq p, q < \infty$, define the *mixed-norm Lebesgue space* $L^p_x L^q_y([0,1] \times [0,1])$ to be the space of all measurable functions $f : [0,1] \times [0,1] \to \mathbf{R}$ whose norm

$$\|f\|_{L^p_x L^q_y([0,1]\times[0,1])} := (\int_0^1 (\int_0^1 |f(x,y)|^q \, dy)^{p/q} \, dx)^{1/p}$$

(or more succinctly, $\|f\|_{L^p_x L^q_y} = \| \|f(x,\cdot)\|_{L^q_y} \|_{L^p_x}$) is finite (with the usual convention that two functions are identified if they agree almost everywhere). You may assume without proof that $L^p_x L^q_y([0,1] \times [0,1])$ is a Banach space.

(a) Show that $L^\infty([0,1] \times [0,1])$ is dense in $L^p_x L^q_y([0,1] \times [0,1])$. [*Hint*: the monotone or dominated convergence theorems may be useful, but please invoke them correctly!]

(b) Establish the inequality $\|f\|_{L^p_x L^q_y([0,1]\times[0,1])} \leq \|f\|_{L^{\max(p,q)}([0,1]\times[0,1])}$ for all $f \in L^{\max(p,q)}([0,1]\times[0,1])$. [Of course, we give $[0,1]\times[0,1]$ the usual product measure. [*Hint*: Hölder's inequality will be useful.]

(c) Show that the space $C([0,1]\times[0,1])$ of continuous functions $f : [0,1]\times[0,1] \to \mathbf{R}$ is dense in $L^p_x L^q_y([0,1]\times[0,1])$.

Problem 2. Let W be a closed subspace of a Banach space V, and let $\iota : W \to V$ be the inclusion map (thus $\iota(x) = x$ for all $x \in W$).

(a) Show that the transpose (or adjoint) map $\iota^* : V^* \to W^*$ is surjective.

(b) Now suppose that V is a real Hilbert space, and identify $V^* \equiv V$ and $W^* \equiv W$ using the Riesz representation theorem for Hilbert spaces. Show that $\iota^* : V \to W$ is the orthogonal projection map (i.e. $\iota^*(x)$ is the orthogonal projection of x to W for each $x \in V$).

Problem 3. Let $T : X \to Y$ be a continuous surjective linear transformation between Banach spaces, and let $T^* : Y^* \to X^*$ be the transpose (or adjoint) map.

(a) Show that there exists a constant $c > 0$ such that $\|T^*\lambda\|_{X^*} \geq c\|\lambda\|_{Y^*}$ for all $\lambda \in Y^*$.

(b) Show that $T^*(Y^*)$ is a closed subspace of X^*.

Problem 4. Let H be a separable Hilbert space, and let $P_1, P_2, \ldots : H \to H$ be a sequence of orthogonal projections (thus for each P_j there exists a closed subspace V_j such that $P_j(x)$ is the orthogonal projection of x to V_j for each $j \geq 1$ and $x \in H$).

(a) Show that there exists a subsequence P_{j_i} of the P_j which converge in the weak operator topology to a bounded self-adjoint linear transformation $T : H \to H$ of operator norm at most 1.

(b) Suppose that the transformation T in (a) is also an orthogonal projection. Show that the P_{j_i} now converge to T in the *strong* operator topology. [*Hint*: if T is an orthogonal projection, then $\langle Tx, x \rangle = \|Tx\|^2$.]

Problem 5. Let $\phi : [0,1] \to [0,1]$ be an increasing diffeomorphism (thus ϕ is smooth and bijective, the inverse is also smooth, and ϕ' is strictly positive). Let E be a Borel set in $[0,1]$. Show that the Lebesgue measure $m(\phi(E))$ of $\phi(E)$ is given by the formula $m(\phi(E)) := \int_E \phi'(x)\ dx$. [*Hint*: There at least three ways to do this; an approach based on approximating Borel sets by finite unions of intervals; an approach based on the Carathéodory extension theorem; and an approach based on the Riesz representation theorem. You may of course use the change of variables formula from undergraduate calculus, as long as the conditions for that formula to apply are met.]

Problem 6. Let K be a closed convex subset of a normed real vector space V, and let x be a point outside of K. Show that there exists a continuous linear functional $\lambda : V \to \mathbf{R}$ and a real number c such that $\lambda(x) > c$, and that $\lambda(y) \leq c$ for all $y \in K$. [*Hint*: It is difficult to apply the Hahn-Banach theorem directly. However, the *proof* of that theorem can be applied here.]

Problem 7. Let $\mu_1, \mu_2, \dots$ be a sequence of Borel probability measures on the real line $\mathbf{R}$. Assume that this sequence of measures is *tight*, which means that for every $\varepsilon > 0$ there exists a compact set $K \subset \mathbf{R}$ such that $\mu_n(\mathbf{R} \backslash K) < \varepsilon$ for all n. Show that there is a subsequence μ_{n_j} of Borel probability measures which converges in the vague topology to another probability measure μ (i.e. $\int_{\mathbf{R}} f \; d\mu_{n_j} \to \int_{\mathbf{R}} f \; d\mu$ for all $f \in C_c(\mathbf{R} \to \mathbf{R})$).