HAMILTONIANS

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At first glance, the many theories and equations of modern physics exhibit a bewildering diversity: compare for instance classical mechanics to quantum mechanics, non-relativistic physics to relativistic physics, or particle physics to statistical mechanics. However, there are strong unifying themes connecting all of these theories. One of these is the remarkable fact that in all of these theories, the evolution of a physical system over time (as well as the steady states of that system) is largely controlled by a single object, the *Hamiltonian* of that system, which can often be interpreted as describing the total energy of any given state in that system. Roughly speaking, each physical phenomenon (e.g. electromagnetism, atomic bonding, particles in a potential well, etc.) may correspond to a single Hamiltonian H, while each type of mechanics (classical, quantum, statistical, etc.) corresponds to a different way of using that Hamiltonian to describe a physical system. For instance, in classical physics, the Hamiltonian is a function $(q, p) \mapsto H(q, p)$ of the positions q and momenta p of the system, which then evolve according to Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}; \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

while in (non-relativistic) quantum mechanics, the Hamiltonian H becomes a linear operator (which is often a formal combination of the position operators q and momenta operators p), and the wave function ψ of the system then evolves by the Schrödinger equation

$$i\hbar \frac{d}{dt}\psi = H\psi.$$

In statistical mechanics, the Hamiltonian H is a function of the microscopic state (or *microstate*) of a system, and the probability that a system at a given temperature T will lie in a given microstate is proportional to $e^{-H/kT}$. And so on and so forth.

Many fields of mathematics are closely intertwined with their counterparts in physics, and so it is not surprising that the concept of a Hamiltonian also appears in pure mathematics. For instance, motivated by classical physics, Hamiltonians (as well as generalisations of Hamiltonians, such as *moment maps*) play a major role in dynamical systems, differential equations, Lie group theory, and symplectic geometry. Motivated by quantum mechanics, Hamiltonians (as well as generalisations, such as *observables* or *pseudo-differential operators*) are similarly prominent in operator algebras, spectral theory, representation theory, differential equations, and in microlocal analysis.

Because of their ubiquitious presence in many areas of physics and mathematics, Hamiltonians are useful for building bridges between seemingly unrelated fields, for instance in connecting classical mechanics to quantum mechanics, or between symplectic mechanics and operator algebras. The properties of a given Hamiltonian often reveal much about the physical or mathematical objects associated to that Hamiltonian; for instance, the symmetries of a Hamiltonian often induce corresponding symmetries on objects described using that Hamiltonian. While not every interesting feature of a mathematical or physical object can be read off directly from its Hamiltonian, this concept is still fundamental to understanding the properties and behavior of such objects.

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