Perelman's proof of the Poincaré conjecture

Terence Tao

University of California, Los Angeles

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- In a series of three papers in 2002-2003, Grisha Perelman solved the famous Poincaré conjecture:
- **Poincaré conjecture** (1904) Every smooth, compact, simply connected three-dimensional manifold is homeomorphic (or diffeomorphic) to a three-dimensional sphere S^3.
- (Throughout this talk, manifolds are understood to be without boundary.)

- The main purpose of this talk is to discuss the proof of this result.
- However, as a warm up, I'll begin with the simpler (and more classical) theory of two-dimensional manifolds, i.e. surfaces.

- **Caution:** This will be a very ahistorical presentation of ideas: results will not be appearing in chronological order!
- Also, due to time constraints, we will **not** be surveying the huge body of work on the Poincaré conjecture, focusing only on those results relevant to Perelman's proof. In particular, we will not discuss the important (and quite different) results on this conjecture in four and higher dimensions (by Smale, Freedman, etc.).

Scalar curvature

- Let M be a smooth compact surface (not necessarily embedded in any ambient space).
- If one gives this surface a Riemannian metric g to create a Riemannian surface (M, g), then one can define the scalar curvature $R(x) \in \mathbf{R}$ of the surface at any point $x \in M$. One definition is that the area of an infinitesimal disk $B(x, r)$ of radius r centred at x is given by the formula

$$\text{area}(B(x,r)) = \pi r^2 - R(x)\pi r^4/24 + o(r^4).$$

Model geometries

If $R(x)$ is independent of x, we say that M is constant curvature. There are three model geometries that have constant curvature:

- The round sphere S^2 (with constant curvature $+1$);
- The Euclidean plane $\mathbf{R}^2$ (with constant curvature 0); and
- The hyperbolic plane H^2 (with constant curvature -1).

- One can create further constant curvature surfaces from a model geometry by rescaling the metric by a constant, or by quotienting out the geometry by a discrete group of isometries.
- It is not hard to show that all connected, constant-curvature surfaces arise in this manner. (The model geometry is the universal cover of the surface.)

Uniformisation theorem

- A fundamental theorem in the subject is
- **Uniformisation theorem** (Poincaré, Koebe, 1907) Every compact surface M can be given a constant-curvature metric g.
- As a corollary, every (smooth) connected compact surface is diffeomorphic (and homeomorphic) to a quotient of one of the three model geometries. This is a satisfactory topological classification of these surfaces.

- Another corollary of the uniformisation theorem is
- **Two-dimensional Poincaré conjecture**: Every smooth, simply connected compact surface is diffeomorphic (and homeomorphic) to the sphere S^2.

Ricci flow

- There are many proofs of the uniformisation theorem, for instance using complex analytic tools such as the Riemann mapping theorem. But the proof that is most relevant for our talk is the proof using Ricci flow.
- The scalar curvature $R = R(x)$ of a Riemannian surface (M, g) can be viewed as the trace of a rank two symmetric tensor, the Ricci tensor Ric. One can view this tensor as a directional version of the scalar curvature; for instance, the area of an infinitesimal sector of radius r and infinitesimal angle θ at x in the direction v is equal to

$$\frac{1}{2}\theta r^2 - \frac{1}{24}\theta r^4 \mathrm{Ric}(x)(v, v) + o(\theta r^4).$$

- In two dimensions, the Ricci tensor can be determined in terms of the scalar curvature and the metric g by the formula

$$\mathrm{Ric} = \frac{1}{2} R g,$$

but this identity is specific to two dimensions.

Ricci flow

- Now introduce a time parameter $t \in \mathbf{R}$. A time-dependent metric $g = g_t$ on a (fixed) manifold M is said to obey the Ricci flow equation if one has

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric},$$

thus positively curved regions of the manifold shrink (in a geometric sense), and negatively curved regions expand. However, the topological structure of M remain unchanged.

A key example

- The two-dimensional sphere in $\mathbf{R}^3$ of radius $r > 0$ is isometric to the standard sphere S^2 with metric $\frac{r^2}{2}g$, and constant scalar curvature $\frac{2}{r^2}$.
- Up to isometry, Ricci flow for such spheres corresponds to shrinking the radius $r = r(t)$ by the ODE $\frac{dr}{dt} = -1/r$, so $r(t) = (r(0)^2 - 2t)^{1/2}$. Thus the Ricci flow develops a singularity at time $r(0)^2/2$, when the radius approaches zero.

- More generally, standard PDE methods give the following local existence result: given any Riemannian surface $(M, g(0))$, a unique Ricci flow $t \mapsto (M, g(t))$ exists for a time interval $t \in [0, T_*)$, with $0 < T_* \leq +\infty$. If $T_* < \infty$, then the curvature blows up (diverges to infinity in sup norm) as $t \to T_*$.
- But suppose one renormalises the surface as it blows up, for instance dilating the metric by a scalar to keep the total area constant. What happens to the geometry in the limit?

- Intuitively, Ricci flow makes a surface rounder, by shrinking the high-curvature "corners" of a manifold at a faster rate than the flatter regions. One way to formalise this intuition is to see that the scalar curvature R for a Ricci flow on a surface obeys the nonlinear heat equation

$$\partial_t R = \Delta R + 2R^2$$

and then to use the maximum principle.

- This suggests that Ricci flow could be used to give a dynamical proof of the uniformisation theorem, if the final state of the (renormalised) Ricci flow has constant curvature.
- Indeed, one has the following theorem:
- **Theorem** (Hamilton 1988, Chow 1991, Chen-Lu-Tian 2005) If (M, g) is topologically a sphere, then Ricci flow becomes singular in finite time, and at the blowup time, the renormalised manifold becomes constant curvature.

- There is a similar theorem in higher genus.
- This theorem easily implies the uniformisation theorem. (Conversely, the uniformisation theorem was used in the original arguments of Hamilton and Chow, but this was removed in Chen-Lu-Tian, thus giving an independent proof of this theorem.)

To oversimplify enormously, the proof proceeds in three stages:

- **Finite time extinction**: any sphere undergoing Ricci flow will blow up in finite time.
- **Rescaled limit**: When one rescales the Ricci flow around the blowup time (e.g. to keep the curvature bounded) and takes limits, one obtains a special type of Ricci flow, a gradient shrinking Ricci soliton (in which the effect of renormalised Ricci flow is equivalent to a diffeomorphism of the coordinates).
- **Classification**: One shows that the only gradient shrinking solitons in two dimensions are the constant curvature surfaces.

Model geometries

Now we return to three dimensions. We again have three constant-curvature model geometries:

- The round sphere S^3 (with constant curvature $+1$);
- The Euclidean space $\mathbf{R}^3$ (with constant curvature 0); and
- The hyperbolic space H^3 (with constant curvature -1).

However, one can also now form model geometries from Cartesian or twisted products of lower-dimensional geometries:

- The product $S^2 \times \mathbf{R}$;
- The product $H^2 \times \mathbf{R}$;
- The (universal cover of) $SL_2(\mathbf{R})$ (a twisted bundle over H^2);
- The Heisenberg group (a twisted bundle over $\mathbf{R}^2$); and
- The three-dimensional solvmanifold (a twisted torus bundle over S^1).

These are the eight Thurston geometries.

Geometrisation conjecture

- One can make more geometries by dilating these model geometries, and quotienting by isometries. However, in three dimensions one also needs to be able to glue several geometries together along spheres S^2 or tori T^2.
- Thurston's geometrisation conjecture (1982) states, among other things, that every smooth oriented compact three-dimensional manifold can be formed from a finite number of these operations, i.e. by gluing together finitely many model geometries or their dilates and quotients.

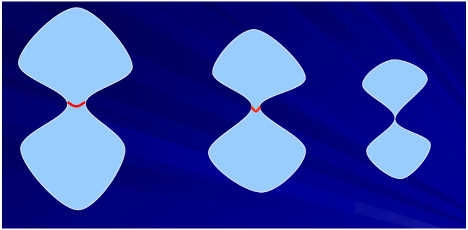
- Thurston verified this conjecture for a large class of manifolds known as Haken manifolds.
- The geometrisation conjecture implies the three-dimensional Poincaré conjecture as a corollary, though this “elliptic” case of the conjecture is not covered by the “hyperbolic” theory of Haken manifolds.
- In the 1980s, Hamilton initiated the program of using Ricci flow to “geometrise” an arbitrary manifold and thus prove the full geometrisation conjecture. This program was essentially completed by Perelman’s three papers, with full details and alternate proofs subsequently appearing in the works of Kleiner-Lott, Bessires-Besson-Boileau-Maillot-Porti, Morgan-Tian, and Cao-Ge.

- In 1982, Hamilton established short-time existence of the Ricci flow in all dimensions, and showed that if the flow could only be continued for a finite time, then the curvature blew up (in sup norm) at that time.
- In the same paper, Hamilton showed that if the Ricci curvature of a three-dimensional manifold was initially positive, then one had finite time blowup, and after rescaling, the limiting manifold had constant curvature (and must then be a quotient of S^3). In particular, this establishes the Poincaré conjecture for manifolds which admit a metric of positive Ricci curvature.

- The positivity of curvature was used in an essential way (in conjunction with a sophisticated version of the maximum principle).
- However, a variant of the argument shows that any limiting blowup profile of a Ricci flow has to have non-negative curvature (the Hamilton-Ivey pinching phenomenon). Intuitively, the point is that negatively curved regions of the flow expand rather than contract, and so do not participate in the blowup profile.

When one does not assume initial positive curvature, the situation becomes more complicated, for two reasons:

- The Ricci flow may not blow up at all. (For instance, on a manifold H^3/Γ of constant negative curvature, the flow simply expands the manifold without ever blowing up.)
- Secondly, even when blowup occurs, the blowup may be localised to a small portion of the manifold (as is the case in a neck pinch singularity). Because of this, the rescaled limiting profile of the flow at the blowup point can be non-compact (e.g. a cylinder $S^2 \times \mathbf{R}$, in the case of a neck pinch) and fail to describe the asymptotic behaviour of the entire manifold at the blowup time.



A neckpinch (John Lott, 2006 ICM)

The first difficulty (lack of singularity formation in finite time) can be handled by working with minimal spheres - minimal surfaces in the manifold diffeomorphic to S^2.

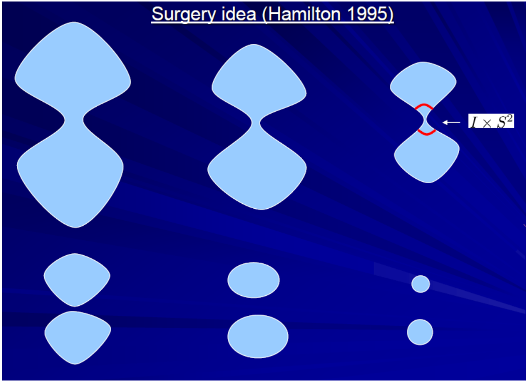
- The Sacks-Uhlenbeck theory of minimal surfaces guarantees that such minimal spheres exist once $\pi_2(M)$ is non-trivial.
- Using Riemannian geometry tools such as the Gauss-Bonnet theorem, one can show that the area of such a minimal sphere shrinks to zero in finite time under Ricci flow, thus forcing a singularity to develop at or before this time.

- This argument shows that finite time singularity occurs unless $\pi_2(M)$ is trivial.
- More sophisticated versions of this argument (Perelman 2003; Colding-Minicozzi 2003) also forces singularity unless $\pi_3(M)$ is trivial.
- Algebraic topology tools such as the Hurewicz theorem show that $\pi_2(M)$ and $\pi_3(M)$ cannot be simultaneously trivial for a compact, simply connected manifold, and so one has finite time singularity development for the manifolds of interest in the Poincaré conjecture. (The situation is more complicated for other manifolds.)

- There is an important strengthening of these results for **Ricci flow with surgery**, which asserts that a simply connected manifold that undergoes Ricci flow with surgery will disappear entirely (become extinct) after a finite amount of time, and with only a finite number of surgeries.
- This fact allows for a relatively short proof of the Poincaré conjecture (as compared to the full geometrisation conjecture, which has to deal with Ricci flows that are never fully extinct), though the proof is still lengthy for other reasons.

To deal with the second issue (localised singularities), one needs to do two things:

- **Classify** the possible singularities in a Ricci flow as completely as possible; and then
- Develop a **surgery** technique to remove the singularities (changing the topology in a controlled fashion) and continue the flow until the manifold is entirely extinct.



Ricci flow with surgery (John Lott, 2006 ICM)

Singularity classification

- The basic strategy in classifying singularities in Ricci flow is to first “zoom in” (rescale) the singularity in space and time by greater and greater amounts, and then take limits.
- In order to extract a usable limit, it is necessary to obtain control on the Ricci flow which is scale-invariant, so that the estimates remain non-trivial in the limit.
- It is particularly important to prevent collapsing, in which the injectivity radius collapses to zero faster than is predicted by scaling considerations.

- In 2003, Perelman introduced two geometric quantities, the Perelman entropy and Perelman reduced volume, which were scale-invariant, which decreased under Ricci flow, and controlled the geometry enough to prevent collapsing. Roughly speaking, either of these quantities can be used to establish the important
- **Perelman non-collapsing theorem** (Informal statement) If a Ricci flow is rescaled so that its curvature is bounded in a region of spacetime, then its injectivity radius is bounded from below in that region also.
- Thus, collapsing only occurs in areas of high curvature.

Perelman entropy

- The heat equation is the gradient flow for the Dirichlet energy, and thus decreases that energy over time. It turns out to similarly represent Ricci flow (modulo diffeomorphisms) as a gradient flow in a number of ways, leading to a number of monotone quantities for Ricci flow. Perelman cleverly modified these quantities to produce a scale-invariant monotone quantity, the Perelman entropy, which is related to the best constant in a geometric log-Sobolev inequality.

Perelman reduced volume

- The Bishop-Gromov inequality in comparison geometry asserts, among other things, that if a Riemannian manifold M has non-negative Ricci curvature, then the volume of balls $B(x, r)$ grows in r no faster than in the Euclidean case (i.e. the Bishop-Gromov reduced volume $\mathrm{Vol}(B(x, r))/r^d$ is non-increasing in d dimensions). Inspired by an infinite-dimensional formal limit of the Bishop-Gromov inequality, Perelman found an analogous reduced volume in spacetime for Ricci flows, the Perelman reduced volume that had similar monotonicity properties.

Limiting solutions

- Using the non-collapsing theorem, one can then extract out a special type of Ricci flow as the limit of any singularity, namely an ancient κ-solution. These solutions exist for all negative times, have non-negative and bounded curvature, and are non-collapsed at every scale.

- To analyse the asymptotic behaviour of these solutions as $t \to -\infty$, Perelman then took a second rescaled limit, using the monotone quantities again, together with some Harnack-type inequalities of a type first introduced by Li-Yau and Hamilton, to generate a non-collapsed gradient shrinking soliton (which are the stationary points of Perelman entropy or Perelman reduced volume).

- These solitons can be completely classified using tools from Riemannian geometry such as the Cheeger-Gromoll soul theorem and Hamilton's splitting theorem, and an induction on dimension. The end result is
- **Classification theorem.** A non-collapsed gradient shrinking soliton in three-dimensions is either a shrinking round sphere S^3, a shrinking round cylinder $S^2 \times \mathbf{R}$, or a quotient thereof.
- There are now several proofs of this basic result, as well as extensions to higher dimensions (Ni-Wallach, Naber, Petersen-Wylie, etc.).

- Using this classification theorem, one can show, roughly speaking, that high-curvature regions of three-dimensional Ricci flows look like spheres, cylinders, quotients thereof, or combinations of these components such as capped or doubly capped cylinders. (The canonical neighbourhood theorem.)
- As a consequence, it is possible to perform surgery to remove these regions. (This is not the case in higher dimensions, when one starts seeing non-removable singularities such as $S^2 \times \mathbf{R}^2$.)

- Surgery methods for Ricci flow were pioneered by Hamilton, but the version of surgery needed for Perelman's argument is extremely delicate as one needs to ensure that all the properties of Ricci flow used in the argument (e.g. monotonicity formulae, finite time extinction results) also hold for Ricci flow with surgery.
- Nevertheless, this can all be done (with significant effort), the net result being that Ricci flow with surgery geometrises any three-dimensional manifold. Running the surgery in reverse, this establishes the geometrisation conjecture, and in particular the Poincaré conjecture as a special (and simpler) case.